Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 58-12 (2024) 191–196

Analytical fractional reduced-order model identification method for processes with overdamped and underdamped response

Juan J. Gude * **Flavia B. Baraldi** ** **Ibon Oleagordia** *
Pablo García Bringas *

* *University of Deusto, Bilbao, Spain (e-mail: {jgude,ibruiz,pablo.garcia.bringas}@deusto.es).*
** *Federal Institute of Santa Catarina, Joinville, Brazil (e-mail: flavia.bb@aluno.ifsc.edu.br)*

Abstract: In this paper, an analytical procedure to identify a fractional-first order plus dead time (FFOPDT) model is presented. This procedure is applied to processes with overdamped or underdamped step response, i.e., fractional order values in the range $0.5 \leq \alpha \leq 2.0$ can be estimated. The required process information is obtained through a simple open-loop step experiment and is based on fitting three arbitrary points (x_1-x_2-x_3%) on the process reaction curve. Some numerical examples are proposed to show the effectiveness and applicability of this procedure for the identification of fractional-order models. The main feature of this approach is that it estimates the FFOPDT model of overdamped and underdamped processes.

Keywords: fractional-order systems, fractional first-order plus dead-time model, process identification, overdamped response, underdamped response.

1. INTRODUCTION

Fractional-order systems have attracted increasing interest in the academic and industrial community in recent years, and there are potential breakthroughs in this field (West (2022); Diethelm et al. (2022); Shah et al. (2023)). Fractional-order systems use fractional calculus, which is the extension of traditional calculus that considers integration and derivation of any order, not just an integer. It is widely recognized by academics and industry professionals that these systems offer several advantages over traditional integer-order systems, including increased accuracy, robustness and higher-order dynamics (Monje et al. (2010)).

In the last four decades, fractional calculus has been successfully employed in the modeling of physical phenomena and systems studied in many fields of science and engineering. These include materials science, electronic devices, theoretical physics, mechanics, etc. A wide spectrum of applications in these or other fields can be found in the texts (Kilbas et al. (2006); Kochubei et al. (2019)).

Another approach is to use fractional-order calculus to model and analyze complex systems. It is essential to note that fractional-order differential equations offer an effective technique for describing many real-world physical systems. Fractional-order models can capture nonlinear and nonlocal dynamics that are difficult to represent using typical integer-order models. To design fractional-order control systems, a reduced-order model of the controlled process is usually required. More specifically, industry requires accurate methods for identifying fractional-order models that are characterized by simplicity of implementation and application, see Gude (2023).

Some identification methods for fractional first-order plus dead-time (FFOPDT) models that have been recently proposed are listed below. Analytical methods based on the process reaction curve proposed in Gude and García Bringas (2022a) and Gude and García Bringas (2022b) offer an effective approach combined with simplicity and ease of application. The procedure outlined in Gude et al. (2023) exploits the asymptotic property of the Mittag-Leffler function for a more accurate estimation of the model's fractional order. Another common approach involves identification methods based on optimization principles (Tepljakov (2017); Guevara et al. (2015); Alagoz et al. (2019); Bourouba et al. (2018)). The hybrid approach detailed in Gude et al. (2024b) combines analytical techniques with those based on optimization, revealing an appropriate balance between the simplicity of the procedure and the accuracy of the identified model. These types of identification procedures will help promote the adoption of this reduced-order fractional model at an industrial level.

The current identification method is based on the analytical procedures to estimate FFOPDT models for processes with overdamped response ($0.5 \leq \alpha \leq 1.0$) proposed in Gude and García Bringas (2022b) and for processes with underdamped response ($1.0 \leq \alpha \leq 2.0$) proposed in Gude et al. (2024a).

The procedure proposed in this paper extends the previous methods of identifying fractional reduced-order models for processes with overdamped or underdamped response, characterized by fractional order values in the range $0.5 \leq \alpha \leq 2.0$. The estimation of the FFOPDT model parameters $\theta = \{K, T, L, \alpha\}$ is performed by considering

10.1016/j.ifacol.2024.08.188

an arbitrary set of points (x_1-x_2-x_3%) on the reaction curve. The effectiveness and applicability of the proposed procedure are verified by simulations of fractional processes with overdamped and underdamped response. This analytical procedure is characterized by a low computational effort compared to other methods, such as those based on optimization techniques.

In this context, the contributions of this paper can be summarized as follows:

(1) The proposed methodology of the FFOPDT model identification procedure has been extended for processes with overdamped and underdamped responses.
(2) Analytical expressions have been derived for the estimation of the FFOPDT model for processes with overdamped and underdamped responses.
(3) The effectiveness and applicability of the proposed identification procedure has been validated by simulation results.

2. OVERVIEW OF THE FRACTIONAL-ORDER MODEL IDENTIFICATION PROCEDURE

This Section outlines the identification procedure for fractional-order models based on three arbitrary points on the process reaction curve.

The fractional reduced-order model used in this work to characterize the dynamics of the industrial process is the FFOPDT. This model is capable of characterizing processes with overdamped and underdamped behavior using the same structure, as illustrated in Muresan and Ionescu (2020).

The transfer function for the FFOPDT model is given by:

$$P(s) = \frac{Ke^{-Ls}}{1 + Ts^{\alpha}} \tag{1}$$

Here, K represents the model's gain, $T > 0$ is the time constant, $L \geq 0$ denotes the apparent dead-time, and α signifies the non-integer order of the model. Note that the standard first-order plus dead-time (FOPDT) model can be considered as a particular case of the fractional-order model (1) when $\alpha = 1.0$.

This paper focuses on processes that exhibit both monotonic and non-monotonic responses in the range of values $\alpha \in [0.5, 2.0]$. The overall procedure for identifying the FFOPDT model, as proposed in Gude and García Bringas (2022b) for processes with overdamped response and, more recently, in Gude et al. (2024a) for processes with underdamped response, is briefly outlined below:

The response of an FFOPDT model to a step-input signal with amplitude Δu is expressed as follows:

$$y_{\alpha}(t) = \begin{cases} 0, & 0 \leq t < L \\ K\left\{1 - E_{\alpha}\left[-\frac{1}{T}(t-L)^{\alpha}\right]\right\}\Delta u, & t \geq L \end{cases} \tag{2}$$

where E_{α} is the so-called one-parameter Mittag-Leffler function, which is defined in Podlubny (1999). Note that the output signal change in eq. (2) can be expressed as $\Delta y = K \cdot \Delta u$.

The process output $y_{\alpha}(t)$ can be normalized to the process output total change Δy and the following change of variable can be applied for replacing the time variable t:

$$\tau = \frac{1}{T}(t-L)^{\alpha}, \tag{3}$$

where τ represents the shifted and normalized time.

Thus, taking into account the above considerations, eq. (2) is reduced to:

$$\tilde{y}_{\alpha}(\tau) = \frac{y_{\alpha}(\tau)}{\Delta y} = 1 - E_{\alpha}(-\tau), \qquad \tau \geq 0 \tag{4}$$

where $\tilde{y}_{\alpha}(\tau)$ depends on α and τ.

From eq. (3), the expression relating t_x and τ_x is as follows:

$$t_x = L + (\tau_x T)^{1/\alpha}, \tag{5}$$

where t_x is the time required to reach the amplitude $y_{\alpha}(t_x)$ on the process reaction curve $y_{\alpha}(t)$, which correspond to the x% of the process output's change, and τ_x is the normalized time required for the normalized process output $\tilde{y}_{\alpha}(\tau)$ to reach x% of the total change in that curve.

The procedure summarized in this Section estimates the model parameters using data information from the process reaction curve, which can be obtained by applying a typical open-loop step-test experiment. Process data required in the procedure are $\{\Delta y, \Delta u, t_{x1}, t_{x2}, t_{x3}\}$. The following set of equations is derived to estimate the parameters of the FFOPDT model:

$$\begin{cases} K = \frac{\Delta y}{\Delta u} \\ \alpha = f_1(\Delta) \\ T = f_2(\alpha)(t_{x3} - t_{x1})^{\alpha} \\ L = max[t_{x3} - f_3(\alpha)T^{1/\alpha}, 0] \end{cases} \tag{6}$$

where t_{x1}, t_{x2}, and t_{x3} are the times required to reach the amplitudes $y_{\alpha}(t_{x1})$, $y_{\alpha}(t_{x2})$, and $y_{\alpha}(t_{x3})$, which correspond to the x_1%, x_2%, and x_3%, respectively, of the process output change. Note that the expressions for the functions $f_1(\Delta)$, $f_2(\alpha)$ and $f_3(\alpha)$ in eq. (6) are determined experimentally by applying curve fitting to data sets, as will be discussed in the next Section. These expressions depend on the selected set of points (x_1-x_2-x_3%) and, hence, on the corresponding normalized times $\{\tau_{x1}, \tau_{x2}, \tau_{x3}\}$ for $0.5 \leq \alpha \leq 2.0$. τ_{x1}, τ_{x2}, and τ_{x3} represent the normalized times required for the normalized process output $\tilde{y}_{\alpha}(\tau)$ to reach x_1%, x_2%, and x_3% of the total change in that curve, respectively.

Figure 1 illustrates an overview of the overall FFOPDT model identification procedure. It is based on fitting three arbitrary points (x_1-x_2-x_3%) on the process reaction curve, which can exhibit either overdamped or underdamped behavior. The procedure is clearly divided into two parts:

Part A (depicted in blue) outlines the algorithm for identifying the FFOPDT model using the data extracted from the process reaction curve. It consists of the following steps:

(1) The process data $\{\Delta y, \Delta u, t_{x1}, t_{x2}, t_{x3}\}$ are obtained from the reaction curve, which may exhibit either overdamped or underdamped behavior.
(2) The value of the ratio index Δ for the process reaction curve is determined as a function of the times t_{x1}, t_{x2}, and t_{x3}, according to the following expression:

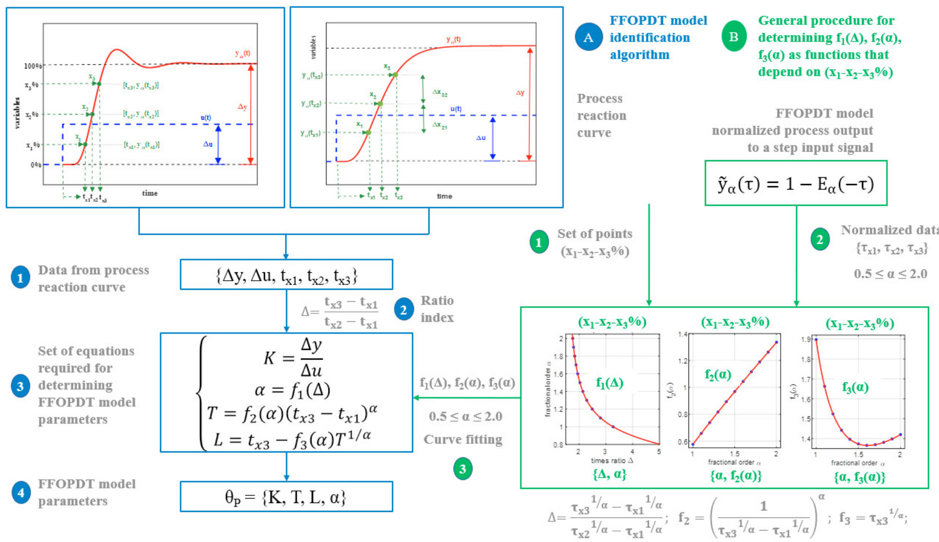

Fig. 1. Overview of the approach presented in this paper for the estimation of an FFOPDT model.

$$\Delta = \frac{t_{x3}^{1/\alpha} - t_{x1}^{1/\alpha}}{t_{x2}^{1/\alpha} - t_{x1}^{1/\alpha}}, \tag{7}$$

(3) The parameters of the FFOPDT model are estimated by applying the set of equations (6). Note that the rational expressions for the functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$ have been previously determined in part B of the overall procedure.
(4) The set of parameters of the FFOPDT model $\theta = \{K, T, L, \alpha\}$ is obtained.

Part B (depicted in green) describes the generic steps to determine rational expressions for the functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$. It consists of the following steps:

(1) The set of points (x_1-x_2-x_3%) on the process reaction curve are selected.
(2) The normalized times $\{\tau_{x1}, \tau_{x2}, \tau_{x3}\}$ are determined for the selected set of points and $0.5 \leq \alpha \leq 2.0$. Note that the values of τ_x can be obtained from the eq. (4).
(3) The expressions for functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$ are experimentally determined by applying curve fitting to the data sets $\{\Delta, \alpha\}$, $\{\alpha, a^\alpha\}$, and $\{\alpha, \tau_{x3}^{1/\alpha}\}$, respectively, for $0.5 \leq \alpha \leq 2.0$.

Functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$, which depend on τ_{x1}, τ_{x2}, τ_{x3}, and α, are used in eq. (6) to estimate the order and time-based parameters of the fractional model:

- Function $f_1(\Delta)$ establishes the relationship between α and Δ from eq. (8), which is defined as:

$$\Delta = \frac{\tau_{x3}^{1/\alpha} - \tau_{x1}^{1/\alpha}}{\tau_{x2}^{1/\alpha} - \tau_{x1}^{1/\alpha}}. \tag{8}$$

- Function $f_2(\alpha)$ is defined as follows:

$$f_2(\alpha) = a^\alpha, \tag{9}$$

where

$$a = \frac{1}{(\tau_{x3}^{1/\alpha} - \tau_{x1}^{1/\alpha})}. \tag{10}$$

- And function $f_3(\alpha)$ is defined as:

$$f_3(\alpha) = (\tau_{x3})^{1/\alpha}. \tag{11}$$

3. SELECTION OF THE ARBITRARY POINTS

As indicated in the previous section, the functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$ are determined by applying curve fitting to the data sets $\{\Delta, \alpha\}$, $\{\alpha, a^\alpha\}$, and $\{\alpha, \tau_{x3}^{1/\alpha}\}$, respectively. These data sets depend on the normalized times τ_{x1}, τ_{x2}, and τ_{x3}, as can be deduced from the eqs. (8) – (11). Therefore, the selection of the set of points (x_1-x_2-x_3%) is essential in identifying the parameters of the FFOPDT model.

For the purposes of this paper, it is sufficient to demonstrate that equations (6) are valid to estimate the parameters of the FFOPDT model for both processes with overdamped and underdamped behavior, using the same identification procedure based on data collected from the process reaction curve.

Nevertheless, the influence of the location for the points (x_1-x_2-x_3%) on the accuracy of the identified fractional-order model is well known (Gude (2023)). For the interested reader, refer to Gude and García Bringas (2022a) and Gude and García Bringas (2022b) for overdamped processes, and Gude et al. (2024a) for underdamped processes. Due to space limitations, the selection of the set of points based on the behavior of the process to be identified will not be analyzed in this work. The set of points selected to be used in this work is (10-65-90%), which has also been used in Gude (2023) in the context of identifying overdamped processes. Therefore, its performance is expected to be significantly better for this type of processes.

Next, the procedure summarized in Figure 1 is followed for the determination of the functions $f_1(\Delta)$, $f_2(\alpha)$ and $f_3(\alpha)$ required for the set of equations (6).

Figure 2 illustrates the data sets $\{\Delta, \alpha\}$, $\{\alpha, a^\alpha\}$, and $\{\alpha, \tau^{1/\alpha}\}$ for (10-65-90%) and $0.5 \leq \alpha \leq 2.0$. The curves for functions $f_1(\Delta)$, $f_2(\alpha)$ and $f_3(\alpha)$, which have been obtained by applying Levenberg-Marquardt algorithm for the least-squares curve fitting, are also included in the figure. The following rational functions have been selected for curve fitting of functions f_1, f_2, and f_3, respectively:

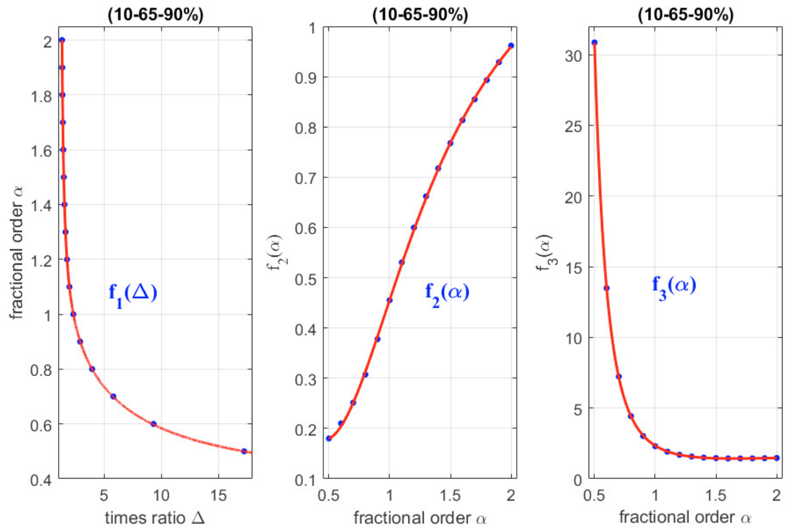

Fig. 2. Data sets considering (10-65-90%), including $\{\Delta, \alpha\}$, $\{\alpha, a^{\alpha}\}$, and $\{\alpha, (\tau_{x3})^{1/\alpha}\}$, for $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$.

$$f_1(\Delta) = \frac{p_1\Delta^2 + p_2\Delta + p_3}{\Delta^2 + q_1\Delta + q_2} \tag{12}$$

$$f_2(\alpha) = \frac{p_1\alpha^2 + p_2\alpha + p_3}{\alpha^2 + q_1\alpha + q_2} \tag{13}$$

$$f_3(\alpha) = \frac{p_1\alpha^2 + p_2\alpha + p_3}{\alpha^2 + q_1\alpha + q_2} \tag{14}$$

Table 1 shows the values of the respective parameters $\{p_i, q_i\}$ associated with the functions f_1, f_2, and f_3, considering the set of points (10-65-90%).

Table 1. Parameters $\{p_i, q_i\}$ for functions $f_1(\Delta)$, $f_2(\alpha)$, and $f_3(\alpha)$, with (10-65-90%).

Parameters	$f_1(\Delta)$	$f_2(\alpha)$	$f_3(\alpha)$
p_1	0.345	1.38	2.472
p_2	4.161	-1.22	-4.321
p_3	-4.228	0.4105	3.00
q_1	2.74	-0.6141	-0.6101
q_2	-4.475	0.866	0.1023

4. SIMULATION RESULTS

In this Section, the effectiveness of the proposed FFOPDT model identification procedure has been tested using two fractional-order process models, one exhibiting over-damped and the other underdamped response. The models considered are of higher order to provide some modeling error or deviation from the FFOPDT dynamics, which is the model structure selected in the proposed identification method.

The models estimated by the proposed method have been compared in terms of accuracy with those obtained with various FFOPDT model identification methods for processes with overdamped and underdamped behavior. The results show that it is possible to use the proposed analytical procedure to identify both overdamped and underdamped processes.

4.1 Example 1

In this example, we consider the model (15), which exhibits fractional dynamics dominated by higher-order lag.

$$P_1(s) = \frac{3}{(1 + 3s^{0.88})(1 + 2s^{0.88})(1 + s^{0.88})} \tag{15}$$

This fractional process has been previously considered in Tavakoli-Kakhki et al. (2010).

The process data summarized in Table 2 shows the information required by the proposed identification procedure.

Table 2. Process data for the process P_1.

Process data for P_1
$\Delta u = 1.00$
$\Delta y = 3.00$
$t_{10} = 2.0290$ s
$t_{65} = 8.8640$ s
$t_{90} = 20.8030$ s

Table 3. FFOPDT model parameters for the process P_1 obtained using the proposed method and other considered methods.

Proposed	Gude (10-50-90%)	Tavakoli-Kakhki
$K_{1,1} = 3.00$	$K_{1,2} = 3.00$	$K_{1,3} = 3.00$
$T_{1,1} = 6.05$ s	$T_{1,2} = 6.64$ s	$T_{1,3} = 6.30$ s
$L_{1,1} = 0.93$ s	$L_{1,2} = 1.39$ s	$L_{1,3} = 1.00$ s
$\alpha_{1,1} = 0.925$	$\alpha_{1,2} = 0.947$	$\alpha_{1,3} = 0.920$

Table 3 lists the FFOPDT model parameters estimated using the proposed procedure. The process (15) is also approximated by the FFOPDT models obtained by following the method proposed by Tavakoli-Kakhki in Tavakoli-Kakhki et al. (2010) and the one proposed by Gude in Gude and García Bringas (2022b) for the symmetrical set of points (10-50-90%).

The step response of the proposed approximate model is compared with the reaction curve of the process P_1 and illustrated in Figure 3. The representative points $x_1 = 10\%$, $x_2 = 65\%$, and $x_3 = 90\%$ are also shown in this figure. To assess the accuracy of the estimated fractional

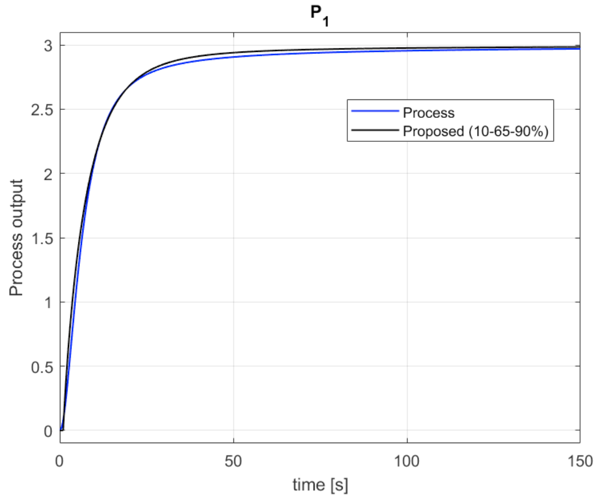

Fig. 3. Comparison between the step response of the FFOPDT model estimated using the proposed identification method and the reaction curve of the process.

models, Table 4 shows the values of the mean squared error (MSE) $S_{1,i}$ $(i = 1, 2, 3)$ of the process P_1 for the different models considered in this example.

Table 4. MSE values $S_{1,i}$, with $(i = 1, 2, 3)$, for the different models estimated for P_1.

Proposed	Gude (10-50-90%)	Tavakoli-Kakhki
$S_{1,1} = 1.9 \cdot 10^{-3}$	$S_{1,2} = 1.3 \cdot 10^{-3}$	$S_{1,3} = 1.4 \cdot 10^{-3}$
Number of samples: $N_S = 1501$		

The results in Figure 3 and Table 4 illustrate that the proposed method is able to identify processes with overdamped response.

The accuracy of the model estimated with the proposed method is similar, although slightly lower, than that obtained using the other two methods.

4.2 Example 2

In this example, we consider the same higher-order fractional model as in the previous example but with distributed fractional orders $\alpha \geq 1$, which provides it with an underdamped behavior.

$$P_2(s) = \frac{3}{(1 + 3s^{1.20})(1 + 2s^{1.20})(1 + s^{1.20})} \tag{16}$$

Table 5 summarizes the process data that have been collected from the reaction curve of the process P_2 and that are necessary to apply the proposed identification procedure. Table 6 provides the FFOPDT model parameters estimated using the proposed procedure, those estimated using Gude's procedure (Gude et al. (2024a)), and those derived from Nie's method (Nie et al. (2016)). These last two procedures are also based on fitting three points on the process reaction curve.

The step response of the proposed approximate model is compared with the reaction curve of process P_2 and illustrated in Figure 4. The representative points $x_1 = 10\%$, $x_2 = 65\%$, and $x_3 = 90\%$ are also shown in this figure.

Table 5. Process data for the process P_2.

Process data for P_2
$\Delta u = 1.00$
$\Delta y = 3.00$
$t_{10} = 2.1710$ s
$t_{65} = 4.9870$ s
$t_{90} = 6.3700$ s

Table 6. FFOPDT model parameters for the process P_2 obtained using the proposed method and other considered methods.

Proposed	Gude (50-65-95%)	Nie (20-60-95%)
$K_{2,1} = 3.00$	$K_{2,2} = 3.00$	$K_{2,3} = 3.00$
$T_{2,1} = 6.55$ s	$T_{2,2} = 4.95$ s	$T_{2,3} = 5.87$ s
$L_{2,1} = 1.28$ s	$L_{2,2} = 1.72$ s	$L_{2,3} = 1.44$ s
$\alpha_{2,1} = 1.496$	$\alpha_{2,2} = 1.406$	$\alpha_{2,3} = 1.462$

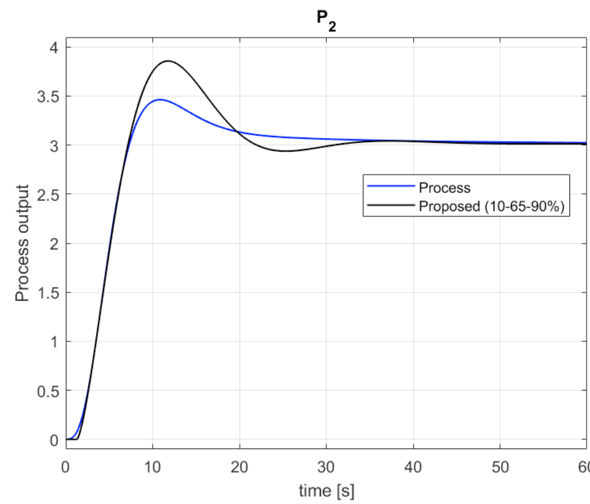

Fig. 4. Comparison between the step response of the FFOPDT model estimated using the proposed identification method and the reaction curve of the process.

To assess the accuracy of the estimated fractional models, Table 7 shows the values of the MSE $S_{2,i}$ $(i = 1, 2, 3)$ of the process P_2 for the different models considered in this example.

Table 7. MSE values $S_{2,i}$, with $(i = 1, 2, 3)$, for the different models estimated for P_2.

Proposed	Gude (50-65-95%)	Nie (20-60-95%)
$S_{2,1} = 1.98 \cdot 10^{-2}$	$S_{2,2} = 4.3 \cdot 10^{-3}$	$S_{2,3} = 1.15 \cdot 10^{-2}$
Number of samples: $N_S = 601$		

The results in Table 7 illustrate that the accuracy of the model estimated with the proposed method is similar to the one obtained using Nie's method. The accuracy of the model proposed by Gude is substantially better than that of the other two.

4.3 Discussion

In this Section, it has been demonstrated that the proposed identification procedure is an analytical method that can estimate FFOPDT models with values $\alpha \in [0.5, 2.0]$. Indeed, the main contribution of this work is that overdamped and underdamped processes with fractional behavior can be identified by an FFOPDT model using a single analytical identification procedure.

Preliminary results are satisfactory for overdamped processes, obtaining FFOPDT models with accuracies in the range of other models estimated by other identification procedures based on the reaction curve. However, there is room for improvement for underdamped processes. For instance, the set of points used in this work provides good results for the overdamped case. Nevertheless, rules for the selection of the set of points that improves the accuracy of the identified model for the underdamped case ($1.0 < \alpha \leq 2.0$) must be explored.

Although there are more accurate FFOPDT model identification procedures in the technical literature, these methods are generally based on optimization and requires a higher computational effort.

Recently, hybrid identification approaches, i.e., those that combine analytical and optimization-based methods to improve the accuracy of the identified model, have been considered (Gude et al. (2024b)). In the authors' opinion, such an approach can achieve improved model accuracy for the underdamped case.

As discussed above, this work focuses on an analytical identification procedure, which is characterized by being simple and requiring low computational effort in its application and hardware implementation. In the authors' opinion, the industry requires this type of identification procedures.

5. CONCLUSION

This paper presents an analytical procedure to identify FFOPDT models. The proposed procedure can be applied to processes with overdamped or underdamped step response, i.e., with fractional order values in the range $0.5 \leq \alpha \leq 2.0$.

The proposed numerical examples have shown the applicability of this procedure for the identification of fractional-order models. More precisely, this identification procedure has provided for the overdamped case results comparable to those obtained with other fractional-order identification methods based on the process reaction curve. However, the procedure requires improvements in the accuracy of the identified model for the underdamped case. Some proposals for improvement have been proposed.

Finally, some conclusions and concluding remarks have been offered in the practical context.

REFERENCES

Alagoz, B.B., Tepljakov, A., Ates, A., Petlenkov, E., and Yeroglu, C. (2019). Time-domain identification of one noninteger order plus time delay models from step response measurements. *International Journal of Modeling, Simulation, and Scientific Computing*, 10(01), 1941011.

Bourouba, B., Ladaci, S., and Chaabi, A. (2018). Reduced-order model approximation of fractional-order systems using differential evolution algorithm. *Journal of Control, Automation and Electrical Systems*, 29(1), 32–43.

Diethelm, K., Kiryakova, V., Luchko, Y., Machado, J.T., and Tarasov, V.E. (2022). Trends, directions for further research, and some open problems of fractional calculus. *Nonlinear Dynamics*, 107(4), 3245–3270.

Gude, J.J. (2023). *Contributions to fractional-order modelling and control of dynamic systems: A theoretical and practical approach*. University of Deusto.

Gude, J.J., Baraldi, F.B., Oleagordia, I., and García Bringas, P. (2024a). Fractional reduced-order model identification method for processes with underdamped response. http://paginaspersonales.deusto.es/jgude/.

Gude, J.J., Di Teodoro, A., Camacho, O., and García Bringas, P. (2023). A new fractional reduced-order model-inspired system identification method for dynamical systems. *IEEE Access*.

Gude, J.J. and García Bringas, P. (2022a). Influence of the selection of reaction curve's representative points on the accuracy of the identified fractional-order model. *Journal of Mathematics*, 2022.

Gude, J.J. and García Bringas, P. (2022b). Proposal of a general identification method for fractional-order processes based on the process reaction curve. *Fractal and Fractional*, 6(9), 526.

Gude, J.J., García Bringas, P., Herrera, M., Rincón, L., Di Teodoro, A., and Camacho, O. (2024b). Fractional-order model identification based on the process reaction curve: A unified framework for chemical processes. *Results in Engineering*, 101757.

Guevara, E., Meneses, H., Arrieta, O., Vilanova, R., Visioli, A., and Padula, F. (2015). Fractional order model identification: Computational optimization. In *2015 IEEE 20th Conference on Emerging Technologies & Factory Automation (ETFA)*, 1–4. IEEE.

Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J. (2006). *Theory and applications of fractional differential equations*, volume 204. elsevier.

Kochubei, A., Luchko, Y., Tarasov, V.E., and Petráš, I. (2019). *Handbook of fractional calculus with applications*, volume 1. De Gruyter Berlin, Germany.

Monje, C.A., Chen, Y., Vinagre, B.M., Xue, D., and Feliu-Batlle, V. (2010). *Fractional-order systems and controls: fundamentals and applications*. Springer Science & Business Media.

Muresan, C.I. and Ionescu, C.M. (2020). Generalization of the fopdt model for identification and control purposes. *Processes*, 8(6), 682.

Nie, Z., Wang, Q., Liu, R., and Lan, Y. (2016). Identification and pid control for a class of delay fractional-order systems. *IEEE/CAA Journal of Automatica Sinica*, 3(4), 463–476.

Podlubny, I. (1999). *Fractional differential equations, mathematics in science and engineering*. Academic press New York.

Shah, P., Sekhar, R., Sharma, D., and Penubadi, H.R. (2023). Fractional order control: A bibliometric analysis (2000–2022). *Results in Control and Optimization*, 100366.

Tavakoli-Kakhki, M., Haeri, M., and Tavazoei, M.S. (2010). Simple fractional order model structures and their applications in control system design. *European Journal of Control*, 16(6), 680–694.

Tepljakov, A. (2017). *Fractional-order modeling and control of dynamic systems*. Springer.

West, B.J. (2022). *Fractional calculus and the future of science*. MDPI, Basel.