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Almost periodic turnpike phenomenon for time-dependent systems

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ABSTRACT

This paper examines the turnpike behavior of a controlled system with almost periodic inputs and a cost functional that includes an almost periodic tracking term and periodic observations. Roughly speaking, we establish that the optimal state–control pair for the controlled system remains exponentially close to the optimal pair of an almost periodic problem for a substantial portion of the time during which the control system evolves. Our approach heavily relies on a thorough investigation of differential Riccati equations, periodic differential Riccati equations, and almost periodic functions.

1. Introduction

1.1. Motivation

In this article, we are interested in the so-called turnpike property, which roughly speaking, describes that the optimal evolutionary solution consists of three arcs: the first and the last being transient short time arcs, and the middle being a long-time arc staying exponentially close to the optimal steady-state, referred to as the turnpike, of the corresponding static optimal control problem. In this scenario, not only the optimal state and control but also the corresponding adjoint vectors remain exponentially close to the stationary optimal control, state and adjoint vectors for a sufficiently large time horizon. The optimal control problem we investigated, in the context of the turnpike phenomenon, is a linear quadratic optimal tracking-control problem.

We will now highlight some of the principal works, to the best of our knowledge, in this field. In the case of finite-dimensional systems, the exponential turnpike property was established under the Kalman rank condition in [1]. For the nonlinear finite dimensional setting we refer to [2]. In [1], a rigorous analysis of the extremal equations has been done for linear infinite dimensional systems under appropriate observability assumptions. These results were extended to semilinear heat equations in [3]. Furthermore, [4] extended these findings for parabolic problems to the Navier–Stokes equations in the two-dimensional setting. All three works [1,3,4] demonstrated the exponential turnpike property by decoupling the extremal equations associated with an appropriate minimization problem, either through the synthesis problem or the Dynamic Programming methodology (see, for instance, [5, Chapter IV]), and by utilizing the algebraic Riccati equation associated with this decoupling.

In [6], the authors established the exponential turnpike property without relying on the Riccati theory but under assumptions of stabilizability and detectability. All the aforementioned works considered optimal control problems without terminal costs, with the exception of [7], which studied terminal costs. Terminal conditions on the state were considered in [8] under controllability assumptions. Additionally, other contributions, though not exhaustive, focusing on turnpike analysis are presented in [9–12]. Lastly, [8] provides a turnpike analysis for general evolution equations, while [13] explores the turnpike property in the context of fractional parabolic equations with exterior controls.

Finally, we want to mention the work [14], where the authors have investigated the turnpike property in Hilbert spaces by using general semigroups method with bounded controls and observation operators, and for parabolic equations with control supported on the boundary. They also study the case of a periodic tracking trajectory, which implies that the referred turnpike is not a stationary problem. Instead of a static problem, a periodic optimal control problem is considered. To the best of our knowledge, this phenomenon, as considered in [14], is the first time that an evolutionary problem is regarded as a turnpike. We would also like to mention the recent article [15], where the exponential turnpike property was studied in the context of a stochastic optimal control problem with periodic coefficients. Specifically, they extended the existing literature on the turnpike property by incorporating a stochastic differential equation with periodic coefficients. Similar to [14], the turnpike in this case refers to a deterministic optimal

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control that is periodic in a distributional sense. Namely, the turnpike is also a nonstationary problem as in [14].

Inspired by the previous studies on periodic tracking problem and the periodic stochastic optimal control, this article aims to investigate the exponential turnpike phenomenon for a class of time-dependent state equations with periodic inputs: periodic control operators and with a linear quadratic cost with periodic observations; and almost periodic controls, source terms and the cost featuring an almost periodic trajectory. Our main result establishes that solutions to the evolutionary extremal equations are exponentially close to their corresponding almost periodic counterparts. In other words, we establish an exponential turnpike property with an almost periodic turnpike problem, enabling the analysis of more general differential equations, as detailed below. The main assumptions we make revolve around the exponential stabilizability and detectability conditions for both the dynamics and the cost functional.

Our proof of the main result is based on a detailed analysis of the almost periodic optimal control problem. Specifically, we exploit the uniform asymptotic stability of solutions to the differential Riccati equation associated with the synthesis problem for the extremal equation. Additionally, we utilize the exponential decay rate of the evolution operator generated by a bounded perturbation by the periodic Riccati operator of the original semigroup generator. This property, which is crucial to our main result, not only plays a significant role in the proof but also distinguishes our work from previous studies on the turnpike phenomenon, as noted in [15]. The primary reason is that, typically, previous studies have employed the exponential decay rate of the semigroup perturbed by the solution of an algebraic Riccati equation associated with an infinite time horizon optimal control problem, a well-established approach (see, for instance, [16]).

The motivation for considering periodic control operators stems from well-established results related to the controllability of partial differential equations, where the control region must move to cover the entire domain over which the dynamics evolve. This situation arises when a static control region is insufficient to achieve exact or null controllability properties. Examples illustrating this bad control behavior are, for instance, the viscoelasticity system studied in [17], where the authors established a null controllability result by using a suitable Carleman inequality over the trajectory of the control which can be taken to be determined by the flow generated by some vector field. In addition, this flow must meet certain requirements so that the control region can cover the entire domain in a finite time. Similar strategy was utilized in [18] in the study of the null controllability of some Sobolev–Galpern equations and in [19] for the Moore–Gibson–Thompson equation. For the improved Boussinesq equation analyzed in [20] the authors proved that the system is null controllable in $H^1(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, in a sufficiently large time. Actually, this control time has to be chosen in such a way that the support of the control, which is moving at constant velocity c, can visit all the domain $\mathbb{T}$. Other examples are the local and nonlocal wave equation with structural damping [21,22], wave equation with memory term [23], among others.

On the other hand, an almost periodic inputs like a tracking term in the cost function allows us to consider more general applied situations. A typical example is the function $\sin(t)+\cos(\sqrt{2}t)$ which is a sum of two purely periodic functions but it is not a periodic function. However, every periodic function is an almost periodic ones. That is, almost periodic function is a generalization of periodicity. The concept of almost periodic function was introduced by the mathematician Harald Bohr in [24] and later continued to be studied by various mathematicians, among them, Salomon Bochner. Almost periodic solutions to boundary value problems for systems of partial differential equations that arise in solving certain problems for inhomogeneous media have been investigated in the research articles [25–27]. Concerning the existence and uniqueness of almost periodic solutions of the Navier–Stokes-type equations and other important examples, the reader may consult the reference list of [28]. Another application of this kind of solutions can be founded in [29,30] for applications of almost periodic functions in crystallography.

1.2. Main result

In this section, we present the main mathematical result of this work. Let X, H be two Hilbert spaces, with $X \subset H \subset X'$, where H is identified with its dual, and the embedding is dense.

We now recall the well-known concept of an almost periodic function as introduced in [24].

Definition 1. Let $g : \mathbb{R} \to H$ be a continuous function. Given $\varepsilon > 0$, we call π an ε-period for g if and only if

$$\|g(t+\pi) - g(t)\| \leq \varepsilon, \quad t \in \mathbb{R}. \tag{1}$$

We denote by $\mathcal{V}(g,\varepsilon)$ the set of all ε-periods for g. We will say that g is an almost periodic function if and only if the set $\mathcal{V}(g,\varepsilon)$ is relatively dense in $\mathbb{R}$. That is, there is a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that in every subinterval of length $L(\varepsilon)$, there is an ε-translation number π, which means a number π such that (1) holds. We call L a modulus of almost periodicity for g.

It is important to highlight the frequent occurrence of almost periodic functions in various contexts. For example, functions of the form $g(t) = \sin(t)+\cos(\sqrt{2}t)$, which is a sum of two purely periodic functions, serve as classical examples of almost periodicity and naturally extend the concept of periodic functions.

We will use the notation $AP(\mathbb{R}; H)$ to represent the Banach space encompassing all continuous almost periodic functions on H with the sup norm. It is worth recalling that every almost periodic function is both bounded and uniformly continuous. Furthermore, the space $AP(\mathbb{R}; H)$ constitutes a Banach algebra.

Let $g_1, g_2 \in AP(\mathbb{R}; H)$. Utilizing the fact that H is a Hilbert space, we can define an inner product on $AP(\mathbb{R}; H)$ as follows:

$$\lim_{\Pi\to\infty} \frac{1}{\Pi} \int_{-\Pi}^{\Pi} \langle g_1(t), g_2(t)\rangle_H \, dt.$$

This inner product is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and the corresponding norm is denoted by $\|\cdot\|_{\mathcal{H}}$ (refer to, for instance, [31]). Consequently, let $L^2_{ap}(\mathbb{R}; H)$ be the completion of $AP(\mathbb{R}; H)$ with respect to this inner product. In a similar fashion, we denote by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and the corresponding norm $\|\cdot\|_{\mathcal{U}}$ for the space $AP(\mathbb{R}; U)$, and by $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and the corresponding norm $\|\cdot\|_{\mathcal{V}}$ for $AP(\mathbb{R}; V)$.

Let $T \gg 1$ and let us consider the optimal control problem that follows

$$\min_{u\in L^2(0,T;U)} J^T(u) = \frac{1}{2}\int_0^T \|C(t)(x(t)-x_d(t))\|_V^2 \, dt + \frac{1}{2}\int_0^T \|N^{1/2}(t)u(t)\|_U^2 \, dt, \tag{2}$$

subject to the state equation

$$\begin{cases} x'(t) = Ax(t) + B(t)u(t) + f(t), & t \in (0,T), \\ x(0) = x_0. \end{cases} \tag{3}$$

Here, $B \in L^\infty(\mathbb{R}; \mathcal{L}(U,H))$, $C \in L^\infty(\mathbb{R}; \mathcal{L}(H,V))$, V is a Hilbert space, and $N \in L^\infty(\mathbb{R}; \mathcal{L}(U,U))$ is an invertible positive definite operator. All three are τ-periodic operators, and $A : D(A) \subset H \to H$ generates an analytic C_0 semigroup $\{S(t)\}_{t\geq 0}$ in H and $i\mathbb{R} \subset \rho(A)$. In this article we consider an external input $f \in AP(\mathbb{R}; H)$ and $x_d \in AP(\mathbb{R}; H)$ a given target, both almost periodic functions of modulus L, in the sense of Definition 1.

The direct method in the calculus of variations guarantees the existence of a minimizer u^T of J^T and an optimal state x^T. Besides, the first order optimality condition ensures the existence of $p^T \in C([0,T]; H)$ such that

$$\begin{cases} (x^T)'(t) = Ax^T(t) + B(t)u^T(t) + f(t), \\ (p^T)'(t) = -A^*p^T(t) - C^*(t)C(t)(x^T(t) - x_d(t)), \\ x^T(0) = x_0, \\ p^T(T) = 0, \\ u^T = -N^{-1}B^*p^T. \end{cases} \tag{4}$$

As mentioned in the introduction, our reference turnpike problem is now an evolutionary problem since our cost functional involve several time-dependent terms. In this case, with the tracking term being an almost periodic function, we treat the turnpike as an almost periodic problem. Namely, we consider the almost periodic optimal control problem of modulus L:

$$\min_{u\in L^2_{ap}(\mathbb{R};U)} J^L(u) = \frac{1}{2}\|C(x-x_d)\|^2_{\mathcal{V}} + \frac{1}{2}\|N^{1/2}u\|^2_{\mathcal{U}}, \tag{5}$$

subject to the initial-at-rest state $x = x(t)$ (for more details on this type of solution, see the following section), which solves the following system

$$x'(t) = Ax(t) + B(t)u(t) + f(t). \tag{6}$$

Determining the existence of an admissible solution $u \in L^2_{ap}(\mathbb{R};U)$ for this problem is not straightforward. In [32] the authors state that to obtain an admissible almost periodic control it suffices to assume that the operator $A - BG$, for some $G \in C(\mathbb{R};\mathcal{L}(H,U))$, generates an exponentially stable evolution operator $U_{A-BG}(t,s)$ for $0 \le s \le t$ and $i\mathbb{R} \subset \rho(A)$. In this case, the feedback law

$$u = -Gx + v, \quad v \in L^2_{ap}(\mathbb{R};U),$$

is admissible, and furthermore,

$$x(t) = \int_{-\infty}^{t} U_{A-BG}(t,s)\Big(B(s)v(s) + f(s)\Big)\,ds,$$

represents the unique initially-at-rest solution of (6). Since f and x_d are almost periodic functions, we will see in the next section that the initially-at-rest solution is indeed an almost periodic solution. Therefore, it is reasonable to assume the existence of such an operator G and $i\mathbb{R} \subset \rho(A)$.

The main assumptions, consistent with the above, for our problem are as follows:

- **(H1)** The pair (A, B) is exponentially stabilizable, meaning that there exists a τ-periodic operator $K \in C(\mathbb{R};\mathcal{L}(H,U))$ such that the evolution operator generated by $A_K(t) := A - B(t)K(t)$ is exponentially stable on H.
- **(H2)** The pair (A, C) is exponentially detectable, indicating that there exists a τ-periodic operator $R \in C(\mathbb{R};\mathcal{L}(V,H))$ such that the evolution operator generated by $A_R(t) := A - R(t)C(t)$ is exponentially stable on H.

Under the preceding hypotheses, as elaborated in the next section, the almost periodic optimization problem has an optimal almost periodic control u^L minimizing the functional J^L, and there exists x^L, the associated optimal state.

Our focus is on analyzing the relationship between problems (2)–(3) and (5)–(6) in the following sense:

> Is it possible for the optimal solutions of the two problems to be close enough for a long time?

We now present our main result, commonly referred to in the literature as the exponential turnpike property. The proof draws inspiration from the results in [1].

Theorem 1. *Let $T \gg 1$. Suppose that* (**H1**) *and* (**H2**) *holds. Let (x^T, u^T) and (x^L, u^L) be the optimal solutions of problems (2) and (5), respectively. Then, there exist positive constants $C > 0$ and $\mu > 0$, such that for any $T \gg 1$*

$$\|x^T(t) - x^L(t)\|_H + \|u^T(t) - u^L(t)\|_U$$
$$\le C\Big(e^{-\mu t}\|x_0 - x^L(0)\|_H + e^{-\mu(T-t)}\|p^L(T)\|_H\Big), \tag{7}$$

for every $t \in [0,T]$.

The remainder of the article is structured as follows. Section 2 provides a detailed exploration of the almost periodic cell, specifically addressing problem (5)–(6). The proof of our main result, Theorem 1, is presented in Section 3. In addition, we present a list of observations from our work and an example of application of the main theorem.

2. The almost periodic cell

In this section, we analyze the almost periodic optimal control problem of modulus L, as introduced earlier. That is, we consider

$$\min_{u\in L^2_{ap}(\mathbb{R};U)} J^L(u) = \frac{1}{2}\|C(x-x_d)\|^2_{\mathcal{V}} + \frac{1}{2}\|N^{1/2}u\|^2_{\mathcal{U}}, \tag{8}$$

subject to the initially-at-rest solution $x = x(t)$ of the following system:

$$x'(t) = Ax(t) + B(t)u(t) + f(t). \tag{9}$$

Here, $B \in L^\infty(\mathbb{R};\mathcal{L}(U,H))$, $C \in L^\infty(\mathbb{R};\mathcal{L}(H,V))$, V is a Hilbert space, and $N \in L^\infty(\mathbb{R};\mathcal{L}(U,U))$ is an invertible positive definite operator. All three are τ-periodic operators, and $f \in AP(\mathbb{R};H)$ and $x_d \in AP(\mathbb{R};H)$ are given almost periodic functions of modulus L (see Definition 1). Additionally, $A : D(A) \subset H \to H$ generates an analytic C_0 semigroup $\{S(t)\}_{t\ge0}$ in H and $i\mathbb{R} \subset \rho(A)$.

Following the article [32], we introduce the concept of *initially-at-rest solution* of (9). This definition is to overcome the difficulty of obtaining an almost periodic solution for (9), since in general $x = x(t)$ will be only an element of $L^2_{ap}(\mathbb{R};H)$ and then no initial condition such as $x(0) = x_0$ could be consider. We refer to [32] for a full discussion in this regard.

Definition 2. Let $f \in AP(\mathbb{R};H)$ and $u \in L^2_{ap}(\mathbb{R};U)$. Then $Bu + f$ has a unique representation as

$$[Bu+f](t) = \sum_{k=1}^{\infty} y_k e^{i\lambda_k t}, \quad \lambda_k \in \mathbb{R}, \quad y_k \in H.$$

We call the input u an admissible input if and only if there exists a sequence $\{x_k\}_k \subset D(A)$ such that $\{\|x_k\|\}_k \in l^2$ and $(i\lambda_k I - A)x_k = y_k$ for every $k \in \mathbb{N}$.

An initial-at-rest solution of (9) corresponding to the input $u = u(t)$ and the forcing term $f = f(t)$ is given by

$$x(t) = \sum_{k=1}^{\infty} x_k e^{i\lambda t}.$$

Since A is the generator of the semigroup $\{S(t)\}_{t\ge0}$ such that $i\mathbb{R} \subset \rho(A)$ and the pair (A, B) is exponentially stabilizable, there exists a unique initially-at-rest solution of (9) and is given by (see [32])

$$x(t) = \int_{-\infty}^{t} S(t-s)(B(s)u(s) + f(s))ds. \tag{10}$$

Moving on, since the operator N is positive definite, the last term of the cost functional guarantees that the control u is bounded in $L^2_{ap}(\mathbb{R};U)$. Moreover, the stabilizability and detectability assumptions (**H1**) and (**H2**), respectively, imply that the cost functional is coercive. Therefore, by the direct method in the calculus of variations, the existence of an optimal $u^L \in L^2_{ap}(\mathbb{R};U)$ and optimal x^L initially-at-rest solution for the optimal control problem is obtained, where J^L attains its minimum at u^L (see also [32]). These directly imply that feedback control given by

$$u(t) = -K(t)x(t) + v(t), \quad v \in L^2_{ap}(\mathbb{R};U),$$

is admissible and in this case the unique initially-at-rest solution of (9) can be expressed as

$$x(t) = \int_{-\infty}^{t} U_{A-BK}(t,s)(B(s)v(s) + f(s))ds,$$

where $U_{A-BK}(t,s)$, for $0 \le s \le t$, is the exponentially stable evolution operator generated by the operator $A_K = A - BK$. Let us recall that since the source term f is an almost periodic function, then u^L is an almost periodic admissible optimal control. Therefore, putting all of the above together, we can conclude that x^L and u^L are almost periodic functions of modulus L (see [32] for further details).

In addition, the first-order optimality conditions hold, ensuring the existence of an almost periodic function p^L such that:

$$\begin{cases}(x^L)'(t) = Ax^L(t) + B(t)u^L(t) + f(t),\\ (p^L)'(t) = -A^*p^L(t) - C^*C(x^L(t) - x_d(t)),\\ u^L = -N^{-1}B^*p^L.\end{cases} \tag{11}$$

In this work, however, we require a closed-loop representation of the optimal solution for the almost periodic system. In other words, we want to obtain a feedback control $u^L = Dx^L$, with some appropriate operator D, such that the optimal state x^L can be written as the solution of the closed loop equation

$$(x^L)'(t) = (A - D)x^L(t) + f(t).$$

To achieve this, we employ the Dynamic Programming approach. That is, first we look for a τ-periodic solution Q of the following differential Riccati equation

$$\begin{cases}Q' + A^*Q + QA - QBN^{-1}B^*Q + C^*C = 0,\\ Q(0) = Q(\tau).\end{cases} \tag{12}$$

It is important to note that, since we consider time-dependent operators in the cost functional, the Riccati equation here is a differential equation, unlike the case in autonomous systems, where the solution to an algebraic Riccati equation is sought.

Then, the optimal control is given by the feedback formula (synthesis problem)

$$u^L(t) = -N^{-1}(t)B^*(t)\Big(Q(t)x^L(t) + r(t)\Big),$$

where $r = r(t)$ is the almost periodic mild solution of the backward problem

$$r' + (A^* - QBN^{-1}B^*)r + Qf - C^*Cx_d = 0.$$

Let us start analyzing the meaning of solution for the τ-periodic Riccati equation. Let $C_s([0,\tau];\mathcal{L}(H,H))$ be the set of all mappings $T : [0,\tau] \to \mathcal{L}(H,H)$ such that $T(\cdot)x$ is continuous for any $x \in H$. Let $\Sigma(H)$ be the set of all linear bounded self-adjoint operator on H, that is,

$$\Sigma(H) := \{T \in \mathcal{L}(H,H) \ : \ T = T^*\}.$$

And, by $\Sigma^+(H)$ we denote the set of all linear bounded self-adjoint nonnegative operators. Namely,

$$\Sigma^+(H) := \{T \in \Sigma(H) \ : \ T \geq 0\}.$$

Definition 3. We will say that $Q \in C_s([0,\tau];\Sigma^+(H))$ is a τ-periodic solution of (12) if Q satisfies the following identity

$$\begin{aligned}Q(t)x &= S^*(\tau - t)Q(0)S(\tau - t)x\\ &- \int_t^\tau S^*(s-t)[Q(s)B(s)N^{-1}(s)B^*(s)Q(s)\\ &-C^*(s)C(s)]S(s-t)x\,ds,\ x \in H,\end{aligned} \tag{13}$$

with $Q(0) = Q(\tau)$.

Under the hypotheses (**H1**) and (**H2**), the following result was proved in [33].

Theorem 2. *[33, Theorem 2.1] Suppose that hypotheses* (**H1**) *and* (**H2**) *holds. Then there exists at most one τ-periodic bounded nonnegative solution $Q \in C_s([0,\tau];\Sigma^+(H))$ of the differential Riccati equation (12). Moreover, the evolution operator $U_F(t,s)$, for $0 \leq s \leq t$, generated by the operator $F := A - BN^{-1}B^*Q$ is exponentially stable. That is, there exist constants $C > 0$ and $\mu > 0$ such that:*

$$\|U_F(t,s)\|_{\mathcal{L}(H,H)} \leq Ce^{-\mu(t-s)}, \quad \forall t \geq s. \tag{14}$$

Even more, an important point in the proof of our main finding is that the τ-periodic solution Q is stable in the following sense.

Lemma 3. *[34, Proposition 3.4] Suppose that hypotheses* (**H1**) *and* (**H2**) *holds. Let $P^T \in C_s([0,T];\Sigma^+(H))$ be the unique mild solution of the Riccati equation*

$$\begin{cases}P' + A^*P + PA - PBN^{-1}B^*P + C^*C = 0, \quad t \in (0,T),\\ P(T) = 0.\end{cases} \tag{15}$$

Then, for each $t \in [0,\infty)$ and $x \in H$ we have

$$\lim_{T\to\infty} P^T(t)x = Q(t)x. \tag{16}$$

Now, following the ideas from [5, Chapter IV], we want to decouple the system (11) by the use of Q, the unique τ-periodic solution of (12). Let us write:

$$p^L = Qx^L + r. \tag{17}$$

A formal computation shows us that Q is the unique τ-periodic solution of (12) and r satisfies:

$$r' + (A^* - QBN^{-1}B^*)r + Qf - C^*Cx_d = 0. \tag{18}$$

Then, the following result is well-known in the context of optimal control [32,35]. In order to ensure the existence of almost periodic solutions for this equation, we strongly use the fact that the convolution operator associated with an exponentially stable semigroup maps almost periodic functions to almost periodic functions, but does not map L^2_{ap}-functions to almost periodic functions (see Example 1.3 in [32]). In this sense, we make strong use of the fact that the source term in the equation satisfied by $r = r(t)$ is an almost periodic function, rather than merely an L^2_{ap}-function.

Theorem 4. *Let us assume that* (**H1**) *and* (**H2**) *hold. Then, for every $f \in AP(\mathbb{R};H)$ and $x_d \in AP(\mathbb{R};H)$ there exists a unique $r \in AP(\mathbb{R};H)$ solution of (18). Moreover, r is given by:*

$$r(t) = \int_t^\infty U_F^*(s,t)\Big(Q(s)f(s) - C^*(s)C(s)x_d(s)\Big)ds, \tag{19}$$

where $U_F(t,s)$, for $0 \leq s \leq t$, is the evolution operator generated by the operator $A - BN^{-1}B^*Q$.

Collecting all the previous content, we finally get that the optimal control of (8) is given by:

$$u^L = -N^{-1}B^*(Qx^L + r), \tag{20}$$

where Q is the unique bounded nonnegative τ-periodic solution of (12) and r is the unique almost periodic solution of (18) and

$$J(u^L) = 2\langle r, f\rangle_{\mathcal{H}} - \|N^{-1}B^*r\|^2_{\mathcal{U}}. \tag{21}$$

In addition, the closed-loop system corresponding to the optimal control problem (8) is given by:

$$(x^L)' = (A - BN^{-1}B^*Q)x^L + f - BN^{-1}B^*r. \tag{22}$$

Since the evolution operator generated by F is exponentially stable, we finally get that x^L is

$$x^L(t) = \int_{-\infty}^t U_F(t,s)\Big(f(s) - B(s)N^{-1}(s)B^*(s)r(s)\Big)ds.$$

For a comprehensive treatment of this topic, we refer to [16, Part V, Chapter 1] and [5, Chapter IV].

3. Proof of Theorem 1

Proof. For clarity, we will divide the proof into several steps.

Step 1: First, we recall some well-known consequences of hypotheses (**H1**) and (**H2**) for the optimal control problem (2). Under these hypotheses, we know that there exists a unique bounded nonnegative operator Q^T, which is the solution of the following differential Riccati equation

$$\begin{cases} Q' + A^*Q + QA - QBN^{-1}B^*Q + C^*C = 0, & t \in (0,T) \\ Q(T) = 0. \end{cases} \tag{23}$$

By Theorem 7.1, Part IV in [16] the following closed loop-system governing x^T:

$$\begin{cases} (x^T)' = (A - BN^{-1}B^*Q^T)x^T + f - BN^{-1}B^*r^T, \; t \in (0,T) \\ x^T(0) = x_0, \end{cases}$$

where r^T is the mild solution of

$$\begin{cases} (r^T)' + (A^* - Q^T BN^{-1}B^*)r^T + Q^T f - C^*Cx_d = 0, \\ r^T(T) = 0. \end{cases} \tag{24}$$

Moreover, the optimal control is given by the feedback formula $u^T = -N^{-1}B^*(Q^T x^T + r^T)$.

Step 2: Let (x^T, u^T, p^T) and (x^L, u^L, p^L) represent the optimal solutions of problems (4) and (11), respectively. Then,

$$(x^T - x^L)' = A(x^T - x^L) - BN^{-1}B^*(p^T - p^L).$$

Since the operator $F = A - BN^{-1}B^*Q$, where Q is the unique τ-periodic nonnegative bounded operator solution of (12), is exponentially stable with rate $\mu > 0$, we rewrite the previous equations as follows

$$(x^T - x^L)' = F(x^T - x^L) + BN^{-1}B^*\Big(Q(x^T - x^L) \\ -(p^T - p^L)\Big).$$

Drawing inspiration from the decoupling method presented in the previous section for the almost periodic problem, let us introduce the new variable

$$h(t) + Q^T(t)(x^T(t) - x^L(t)) = p^T(t) - p^L(t), \quad t > 0. \tag{25}$$

Therefore, we get

$$(x^T - x^L)' = F(x^T - x^L) + BN^{-1}B^*(Q - Q^T)(x^T - x^L) \\ -BN^{-1}B^*h \tag{26}$$

and h is the unique solution of the backward problem

$$\begin{cases} h' = -(A^* - Q^T BN^{-1}B^*)h, & t \in (0,T) \\ h(T) = -p^L(T). \end{cases} \tag{27}$$

To prove the exponential estimate for $x^T - x^L$, we must first derive similar estimates for the difference $Q - Q^T$ and for h. These will be the next two steps in our proof and the most important key tool is the exponentially stability of the evolution operator generated by F.

Step 3: In this step we will show that the τ-periodic solution Q of the Riccati equation (12) is exponentially close to the solution Q^T of the differential Riccati equation (23).

Let us consider the Banach space

$$\mathcal{J} := C_\mu([0,\infty); \Sigma(H)) \\ = \{Y \in C_s([0,\infty); \Sigma(H)) : \sup_{t\ge 0} \|e^{2\mu t}Y(t)\| < \infty\}, \tag{28}$$

with its norm denote by $\|\cdot\|_\mu$. In addition, for every $r > 0$ let $\mathcal{B}_r$ the ball of radius $r > 0$ in $\mathcal{J}$, that is,

$$\mathcal{B}_r := \{Y \in C_\mu([0,\infty) : \Sigma(H)) : \sup_{t\ge 0} \|e^{2\mu t}Y(t)\| \le r\}. \tag{29}$$

Now, set $Z \in \mathcal{B}_r$. Then, it is immediate that $Z = Z(t)$ is the mild solution of the following differential equation

$$\begin{cases} Z' + F^*Z + ZF - ZBN^{-1}B^*Z = 0, & t \in (0,T) \\ Z(T) = Z_0 \end{cases} \tag{30}$$

if and only if, the application $\mathcal{T} : \mathcal{B}_r \to \mathcal{B}_r$ defined by

$$\mathcal{T}(Z)(t)x = U_F^*(T,t)Z_0U_F(T,t)x \\ - \int_t^T U_F^*(s,t)Z(s)B(s)N^{-1}(s)B^*(s)Z(s)U_F(s,t)x\,ds,$$

for every $x \in H$, has a fixed point in $\mathcal{B}_r$. We now demonstrate the existence of a fixed point for $\mathcal{T}$. Indeed, for $Z \in \mathcal{B}_r$ we get

$$\|\mathcal{T}(Z)\|_\mu \le \|Z_0\|_\mu + C\|Z\|_\mu^2. \tag{31}$$

Here we use in a strong way that the evolution operator $U_F(t,s)$, for $0 \le s \le t$, generated by F is exponential stable with rate $\mu > 0$.

Chosen $r > 0$ sufficiently small in the sense that $\|Z_0\|_\mu < \frac{r}{2}$ and $\frac{r}{2} + Cr^2 \le r$, we obtain that $\|\mathcal{T}(Z)\|_\mu \le r$. Namely, the set $\mathcal{B}_r$, which is a closed, convex, and bounded set, is invariant with respect to the application $\mathcal{T}$. From the definition of $\mathcal{T}$ is direct that this operator is continuous. The exponentially decay in the norm of the evolution operator, combined with the fact that the integral is taken over a finite interval and involves compact embeddings, guarantees that $\mathcal{T}$ maps bounded sets into relatively compact sets. Hence, $\mathcal{T}$ is compact. Therefore, by Schauder's fixed point theorem, there exists a fixed point $Z \in \mathcal{B}_r$ of $\mathcal{T}$.

Observe the following fact. Since the operator Q is stable in the sense of (16), we know that there exists $t_0 > 0$ such that we can chose $Z_0 = Q(T - t_0) - Q^T(T - t_0)$ and $\|Q(T - t_0) - Q^T(T - t_0)\|_\mu < \frac{r}{2}$. Then, the fixed point satisfies

$$\begin{cases} Z' + F^*Z + ZF - ZBN^{-1}B^*Z = 0, & t \in (0,T) \\ Z(T) = Q(T - t_0) - Q^T(T - t_0). \end{cases} \tag{32}$$

Using the previous result, we are able to prove the following estimate

$$\|Q(t) - Q^T(t)\| \le Ce^{-2\mu(T-t)}, \quad \forall t > 0. \tag{33}$$

Indeed, by performing an argument similar to the one performed to determine (26), we obtain

$$(Q - Q^T)' + F^*(Q - Q^T) + (Q - Q^T)F \\ -(Q - Q^T)BN^{-1}B^*(Q - Q^T) = 0, \quad t \in (0,T). \tag{34}$$

Thus, the fixed point Z satisfies the same equation of $Q - Q^T$. That is, we can deduce that $Z(t) = Q(t - t_0) - Q^T(t - t_0)$. As we have $Z \in \mathcal{B}_r$, we finally get

$$\|Q(t) - Q^T(t)\| \le Ce^{-2\mu(T-t)}, \quad \forall t \in (0,T).$$

Step 4: Now it is time to prove a similar result for h. As in the previous step, it is possible to rewrite the system (27) as follows

$$\begin{cases} h' = -(A - B^*N^{-1}BQ)^*h + (Q^T - Q)BN^{-1}B^*h, \; t \in (0,T) \\ h(T) = -p^L(T). \end{cases} \tag{35}$$

Moreover, we know that the mild solution $h = h(t)$ of (35) is given by

$$h(t) = -U_F^*(T,t)p^L(T) \\ + \int_t^T U_F^*(s,t)(Q^T(s) - Q(s))B(s)N^{-1}(s)B^*(s)h(s)\,ds. \tag{36}$$

Using the exponential decay of the evolution operator $U_F(t,s)$, with $0 \le s \le t$, generated by $F = A - B^*N^{-1}BQ$ and (33), we deduce

$$\|h(t)\| \le e^{-\mu(T-t)}\|p^L(T)\| + C\int_t^T e^{-\mu(s-t)}e^{-2\mu(T-s)}\|h(s)\|\,ds \\ = e^{-\mu(T-t)}\|p^L(T)\| + Ce^{-\mu(T-t)}\int_t^T e^{-\mu(T-s)}\|h(s)\|\,ds.$$

A direct consequence of the Gronwall Lemma gives us

$$\|h(t)\| \le Ce^{-\mu(T-t)}\|p^L(T)\|.$$

Step 5: In the final step, we establish the turnpike property for our problem. Returning to Eq. (26), the previous two steps allows us to obtain the exponential estimate. Indeed, since

$$(x^T - x^L)' = F(x^T - x^L) + BN^{-1}B^*(Q - Q^T)(x^T - x^L) - BN^{-1}B^*h,$$

we have that

$$x^T(t) - x^L(t) = U_F(t,0)(x_0 - x^L(0)) + \int_0^t U_F(t,s)\Big(B(s)N^{-1}(s)B^*(s)(Q(s) - Q^T(s))(x^T(s) - x^L(s)) - B(s)N^{-1}(s)B^*(s)h(s)\Big)ds.$$

Therefore, using the exponential bounds obtained in steps three and four, the following estimate can be done

$$\begin{aligned}\|x^T(t) - x^L(t)\| &\le \|x_0 - x^L(0)\|e^{-\mu t} + C\int_0^t e^{-\mu(t-s)}\Big(e^{-2\mu(T-s)}\|x^T(s) - x^L(s)\| + e^{-\mu(T-s)}\|p^L(T)\|\Big)\,ds\\ &\le \|x_0 - x^L(0)\|e^{-\mu t} + C\|p^L(T)\|e^{-\mu(T-t)} + Ce^{-\mu(T-t)}\int_0^t e^{-\mu(2T-t)}e^{3\mu s}\|x^T(s) - x^L(s)\|ds.\end{aligned}$$

Applying Gronwall inequality one more time, we get the desired estimate for the optimal state

$$\|x^T(t) - x^L(t)\| \le C\Big(\|x_0 - x^L(0)\|e^{-\mu t} + \|p^L(T)\|e^{-\mu(T-t)}\Big),$$

for every $t \in (0,T)$.

Lastly, the required estimate for the control can be easily derived from identity (25). Indeed, since $h(t)+Q^T(t)(x^T(t)-x^L(t)) = p^T(t)-p^L(t)$ and using the fact that Q^T is bounded, we get

$$\begin{aligned}\|u^T(t) - u^L(t)\| &\le C\|p^T(t) - p^L(t)\|\\ &\le C\|h(t)\| + C\|x^T(t) - x^L(t)\|\\ &\le C\Big(\|x_0 - x^L(0)\|e^{-\mu t} + \|p^L(T)\|e^{-\mu(T-t)}\Big),\end{aligned}$$

for every $t \in (0,T)$. □

Remark 1. Let us mention some comments regarding our work:

- Let us observe that the key tool for obtaining the exponential estimate for $Q-Q^T$ and h, solutions of (34) and (35), respectively, is the exponential stability of the evolution operator generated by $F = A - BN^{-1}B^*Q$. In contrast to previous literature, as seen in [1,3,4,7,14], where the solution of an algebraic Riccati equation for the infinite horizon LQ problem was considered, in our case, this operator is given by the τ-periodic solution of a differential Riccati equation.
- In the previous work [14], the authors have studied the exponential turnpike property when the desired target, namely x_d, is a periodic function. They proved the turnpike using the Riccati's theory as well. However, it is worth mentioning some differences with respect to our paper. The first is related to the operators involved in the state-equation and the cost functional. We do not only take into account the almost periodic target, but we can also consider periodic control and observation operators. The second has to do with the use of the Riccati differential equation, both for the finite and infinite horizon problem.
- Additionally, in [15,36], the authors studied the turnpike property for periodic systems and for stochastic periodic systems, respectively. Here, we move one step further by considering almost periodic systems which allows us to study more general systems. It is important to mention that the proof of Step 3 in our article is different from the one performed in [36], where the authors used the Yosida approximation to obtain the exponential closeness of the periodic Riccati solution to the solution of the Riccati differential equation (23).
- An interesting line of future research would be to analyze the possibility of extending the results presented in this article to the stochastic case [15].
- Our referenced turnpike system is an almost periodic control problem in infinite horizon time. To the best of our knowledge, this is the first work considering this type of turnpike. In the same line, we can mention that if the target function x_d and the source term f are both τ-periodic, then we can consider a τ-optimal control problem as a turnpike. That is,

$$\min_{u\in L^2(0,\tau;U)} J^\tau(u) = \frac{1}{2}\int_0^\tau \|C(t)(x(t) - x_d(t))\|_V^2\,dt + \frac{1}{2}\int_0^\tau \|N^{1/2}(t)u(t)\|_U^2\,dt,$$

  subject to the periodic solution $x = x(t)$ of the following system:

$$\begin{cases} x'(t) = Ax(t) + B(t)u(t) + f(t), & t \in (0,\tau),\\ x(0) = x(\tau),\end{cases}$$

  and then extend both the control and state by periodicity to $\mathbb{R}$.
- An important result to highlight is that our exponential estimate for the difference $Q - Q^T$ given in (33) is of order 2μ as in [36], while in [1] a rate equal to μ is established for parabolic problems.
- It is important to mention that also our turnpike estimate has an explicit dependence on the terms $\|x_0 - x^L(0)\|$ and $\|p^L(T)\|$.
- The assumption on the control operator belonging to the space $L^\infty(\mathbb{R};\mathcal{L}(U,H))$ is not restrictive. This is to avoid technicalities when considering the control operator in $L^\infty(\mathbb{R};\mathcal{L}(U,X'))$. If we assume only the last regularity on our control operator, the turnpike result could only be obtained for the adjoint difference, that is, for $\|p^T(t) - p^L(t)\|$ instead of the controls $\|u^T(t) - u^L(t)\|$.

Finally, let us consider the following example.

Example 1. Let $T > 0$ and Ω denote a bounded open subset of $\mathbb{R}^N$ with a C^2 boundary. We introduce the optimal control represented by

$$\inf_{u\in L^2(\Omega\times(0,T))} J^T(u) = \frac{1}{2}\int_0^T \Big(\|y(x,t) - y_d(x,t)\|_{L^2(\Omega)}^2 + \|u(x,t)\|_{L^2(\omega(t))}^2\Big)dt, \tag{37}$$

where $y_d \in C(\mathbb{R};L^2(\Omega))$ is an almost periodic function, subject to $y \in C([0,T];H_0^1(\Omega)) \cap C^1(]0,T];L^2(\Omega))$ satisfying

$$\begin{cases} y_{tt} - \Delta y - \Delta y_t + b(x)y_t = \chi_{\omega(t)}u(x,t), & \text{in } \Omega\times(0,T),\\ y = 0, & \text{on } \partial\Omega\times(0,T),\\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega,\end{cases} \tag{38}$$

where $b \in L^\infty(\Omega)$ is a given function representing frictional damping, and $(y_0,y_1) \in H_0^1(\Omega)\times L^2(\Omega)$ are the initial conditions.

Theorem 1 is applied to this system by rewriting the viscoelasticity system as a first-order Cauchy problem. Let $H = H_0^1(\Omega)\times L^2(\Omega)$ and $U = V = L^2(\Omega)$. Then,

$$\begin{cases} Y_t + AY = B(t)U, & \text{in } (0,T),\\ Y(0) = (y_0,y_1)^\top,\end{cases}$$

where $Y = (y,y_t)^\top$, $U = (0,u)^\top$, and the operators A and $B = B(t)$ are given by

$$A = \begin{pmatrix} 0 & -I\\ -\Delta & -\Delta + bI\end{pmatrix}, \quad B(t) = \begin{pmatrix} 0\\ \chi_{\omega(t)}\end{pmatrix}.$$

Note that A can be seen as a bounded perturbation of A_0, i.e., $A = A_0+P$ where

$$A_0 = \begin{pmatrix} 0 & -I\\ -\Delta & -\Delta\end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0\\ 0 & bI\end{pmatrix}.$$

The operator A_0 is equipped with the natural domain

$$\begin{cases} D(A_0) = (u,v) \in H_0^1(\Omega)\times H_0^1(\Omega): \ u+v\in H^2(\Omega), \\ A_0(u,v) = (-v, -\Delta(u+v)) \in H_0^1(\Omega)\times L^2(\Omega). \end{cases}$$

Observe that P is a bounded operator, as $b \in L^\infty(\Omega)$. Therefore, A_0 is a m-dissipative operator on $H_0^1(\Omega)\times L^2(\Omega)$. The well-posedness of system (38) on $H_0^1(\Omega)\times L^2(\Omega)$ is an immediate consequence of the fact that A is a bounded perturbation of a m-dissipative operator A_0. Standard semigroup theory implies that A itself generates a strongly continuous semigroup on $H_0^1(\Omega)\times L^2(\Omega)$.

In the study conducted by Chavez et al. [17], the authors demonstrated the achievability of null controllability for the system (38) with an interior control by using a moving control region that covers the entire domain where the dynamics evolve. Consequently, the function $\omega(t)$ can be considered as a τ-periodic function, where $\tau > 0$ represents the minimal time required to cover the entire domain with $\omega(t)$.

Assuming that $(\Omega, \omega(t))$ satisfies a set of geometric properties outlined in [17] (refer to equations (1.11)-(1.17)), the authors established in the same work that there exists a time $T_0 > 0$ such that for every $T > T_0$, the null controllability property holds for the solution of (38). Consequently, we can deduce that the system is exponentially stabilizable with respect to (A, B) and exponentially detectable with respect to (A, C). According to Theorem 1, we can assert that the almost periodic turnpike property holds for the viscoelastic system when $T > \tau$.

Applying a similar argument, we can derive a periodic turnpike property for the Barenblatt–Zheltov–Kochina and Benjamin–Bona–Mah ony equations (both Sobolev–Galpern type equations) presented in [18]. In this case, a moving control domain was employed to achieve a null controllability result. This approach is also applicable to the Moore–Gibson–Thompson equation, a third-order partial differential equation studied in [19], where a moving control domain was also utilized. In both articles, the authors referred to the list of geometrical properties satisfied by the control region introduced in [17] as the "Moving Geometrical Control Condition".

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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