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Class of quasi fractional analytic functions ⋆

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Abstract: The document presents a class of quasi-fractional analytic functions, exploring their properties in complex analysis and fractional calculus. Definitions, theorems, and proofs are established that link these functions with concepts such as Gauss's Theorem and the Cauchy-Pompeiu formula. Additionally, the relationship between harmonicity and quasi-fractional analyticity is investigated, also introducing the concept of quasi-generalized fractional analytic functions.

Keywords: Fractional Analytic Functions, Generalized Fractional Analytic Functions, Harmonicity, Quasi Fractional Analytic Functions, Fractional calculus

1. INTRODUCTION

This paper introduces an expansive area of research that merges the fields of complex analysis and fractional calculus, laying the groundwork for a class of quasi-fractional analytic functions and their generalizations in hypercomplex algebras.

The research delves into how the tools and concepts of fractional calculus which extends differentiation and integration operations to non-integer orders can be applied to complex analysis, which traditionally studies functions of complex variables. This approach seeks to explore new properties and applications in domains where classical operators are insufficient.

To achieve this, the work builds upon a solid foundation of Complex analysis, a fundamental branch of mathematics that focuses on functions of complex variables. In this field, a complex-valued function is called analytic over a specific domain if it meets the conditions of the Cauchy-Riemann equation, which essentially ensures its differentiability in the complex plane. Key concepts such as Gauss's Theorem (in its complex form) and the Cauchy-Pompeiu formula are foundational in this area, providing powerful integral representations and revealing fundamental properties for analytic functions, see Begehr (1996, 2005). Functions that are analytic can also be locally expressed through power series. Furthermore, harmonic functions, which are twice continuously differentiable and satisfy a specific partial differential equation related to the Laplacian operator, are closely linked to analytic functions, see Struppa et al. (2012); Tutschke and Vasudeva (2004); Ablowitz and Fokas (2003).

Moving beyond classical paradigms, fractional calculus extends the traditional notions of integration and differentiation to non-integer orders. This field defines concepts such as the Riemann-Liouville fractional integrals and derivatives, which are generalizations of their integer-order counterparts, see Samko et al. (1993); Kilbas et al. (2006); Torres and Malinowska (2012). These definitions pave the way for a theorem analogous to the fundamental theorem of classical calculus, establishing a core relationship between fractional integration and differentiation, see Miller and Ross (1993); Lovoie et al. (1976); Diethelm (2010); Sabatier et al. (2007).

Fractional calculus has seen burgeoning importance due to its wide-ranging applications in diverse fields, including: Physics Such as quantum mechanics, addressing gauge hierarchy problems, studying supersymmetries, and nanoelectronics, see Hilfer (2003). Biology, economics, geophysics, medicine, and bioengineering. Among others, see Magin (2006); Kilbas and Marzan (2005); Alsaedi et al. (2015).

The convergence of these mathematical domains has led to fractional complex calculus, where a fractional Cauchy-Riemann operator is defined, generalizing its classical counterpart. A complex function is then deemed fractional analytic if it satisfies a fractional version of the Cauchy-Riemann equation, see Ceballos et al. (2020); Kähler and Vieira (2014).

This paper introduces a novel class of functions called quasi-fractional analytic functions. A function is defined as quasi-fractional analytic within a given domain if its real and imaginary parts satisfy a specific system of equations involving both classical partial derivatives and Riemann-Liouville fractional derivatives. An example of fractional analytic function is provided within the paper, Coloma et al. (2021); Ceballos et al. (2022); González-Cervantes and Bory-Reyes (2024) .

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One of the paper's central contributions is establishing a fundamental connection: a function is quasi-fractional analytic if and only if a related auxiliary function (constructed using specific fractional integrals of its real and imaginary components) is classically analytic. This means that many well-known properties and theorems from classical complex analysis can be directly applied to these new quasi-fractional functions. For instance, since this auxiliary function is analytic, it consequently possesses a power series representation.

Furthermore, the work investigates the relationship between harmonicity and this new class of analyticity within the framework of fractional calculus. The paper also delves into the class of quasi-generalized fractional analytic functions. An important lemma states that if a function is quasi-fractional analytic, a specific transformation of its auxiliary function results in a generalized analytic function.

Finally, this paper presents a significant theorem by contradiction: it establishes that if a function is classically analytic and non-constant, with its real and imaginary parts being non-constant separable functions, then it cannot be quasi-fractional analytic.

2. PRELIMINARIES

In this section, we will present the basic results needed to develop the article. We will introduce fundamental concepts, definitions, and basic theorems from fractional calculus, complex analysis, and the combination of both topics in complex fractional calculus.

2.1 Fractional Calculus

Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals of order $\alpha > 0$, denoted by $I_x^\alpha[f]$, are defined as follows (see Kilbas et al. (2006); Podlubny (1999); Samko et al. (1993)):

$$I_x^\alpha[f](x) \ = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a, \tag{1}$$

where $\Gamma(\alpha)$ denotes the Gamma function (see Kilbas et al. (2006); Samko et al. (1993)).

In the next theorem, we introduce the class of functions h, $I_x^\alpha(L_1)$, which consists of functions that can be represented as a fractional integral (1) of a summable function, that is, functions of the form $h = I_x^\alpha[\varphi]$, where $\varphi \in L_1(a,b)$. A description of this class of functions can be found in Kilbas et al. (2006); Samko et al. (1993).

Theorem 2.1. A function $h \in I_x^\alpha(L_1), \alpha > 0$, if and only if its fractional integral $I_x^{s-\alpha}[h] \in AC^s([a,b])$, where $s = [\alpha]+1$ and $(I_x^{s-\alpha}[h])^{(k)}(a) = 0$, for $k = 0, \ldots, s-1$.

Proof 1. See Samko, Theorem 2.3 Samko et al. (1993).

Remark 2.2. $AC^s([a,b])$ denotes the class of functions h, which are continuously differentiable on the segment $[a,b]$, up to order $s-1$ and $h^{(s-1)}$ is absolutely continuous on $[a,b]$. Removing the last condition in Theorem 2.1, we get a class of functions that admits a summable fractional derivative.

Definition 2.3. The fractional Riemann–Liouville derivative, $D_x^\alpha[h](x)$, of order $\alpha > 0$, (see Kilbas et al. (2006); Miller and Ross (1993); Podlubny (1999)) is defined by

$$D_x^\alpha[h](x) = \left(\frac{d}{dx}\right)^s I_x^{s-\alpha}[h](x), \qquad s = [\alpha]+1,\ x > a. \tag{2}$$

For $0 < \alpha < 1$, (2) takes the form

$$D_x^\alpha[h](x) \ = \frac{d}{dx}\, \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{h(t)}{(x-t)^\alpha}\, dt. \tag{3}$$

Definition 2.4. A function $h \in L_1(a,b)$ has a summable fractional derivative $D_x^\alpha[h](x)$ if

$$I_x^{s-\alpha}[h](x) \in AC^s([a,b]),$$

where $s = [\alpha]+1$ (see Samko et al. (1993)).

The definitions for fractional integral and differential operators allow for an analogous theorem to the classical fundamental theorem of calculus.

Theorem 2.5. Let $\alpha > 0$. If $f(x) \in L_p(a,b)$, $1 \leqq p \leqq \infty$, then the following identity

$$(D_x^\alpha I_x^\alpha)[f](x) = f(x),$$

is valid almost everywhere on $[a,b]$ (The space $L^p(\Omega)$, for $1 \leq p < \infty$, consists of all measurable functions f such that $\int_\Omega |f(x)|^p\, dx < \infty$).

Proof 2. See Kilbas, Lemma 2.4. Kilbas et al. (2006).

Example 2.6. Coloma et al. (2021); Ceballos et al. (2022) Let $\mu, \lambda \in$ (0,1), $a > 0$, k $\in \mathbb{N}$ and $\beta > -1$, then

(1) $I_x^\lambda\left[(x-a)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\lambda+1)}(x-a)^{\beta+\lambda}$.

(2) $D_x^\mu\left[(x-a)^\beta\right] =$

$$\begin{cases} 0,\ \beta = \mu - 1, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\mu+1)}(x-a)^{\beta-\mu}, \text{ otherwise.} \end{cases}$$

Example 2.7. Let $\lambda \in$ (0,1), $a = 0$, k $\in \mathbb{N}$ and $\beta > -1$, then

$$I_0^{1-\lambda}\left[x^{\lambda-1}\right] = \frac{\pi \csc(\pi\lambda)}{\Gamma(1-\lambda)} \in \mathbb{R}. \tag{4}$$

2.2 Complex Analysis

Let $a = a_0 + ia_1$ and $b = b_0 + ib_1$ be such that $-\infty < a_i < b_i < \infty$ for $i = 0, 1$. We define the domain

$$\Omega := \{x + iy \mid a_0 < x < a_1,\ b_0 < y < b_1\}. \tag{5}$$

The Cauchy–Riemann operator is defined on a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ by

$$\partial_{\overline{z}} f = \frac{1}{2}(\partial_x + i\partial_y)(u + iv) \tag{6}$$

Then, the complex equation $\partial_{\overline{z}} f = 0$ leads to the well-known Cauchy-Riemann system for $f = u + iv$. Here $z = x + iy$ and the conjugate is $\overline{z} = x - iy$

Definition 2.8. A complex-valued function f is called *analytic* over Ω, if it satisfies the Cauchy-Riemann equation given by $\partial_{\overline{z}} f = 0$.

Definition 2.9. A real-valued function f is called *harmonic* over Ω, if it is twice continuously differentiable and satisfies the following partial differential equation

$$\Delta f = f_{xx} + f_{yy} = 0$$

Theorem 2.10. **Gauss Theorem** (complex form) Let $w \in C^1(\Omega;\mathbb{C}) \cap C(\overline{\Omega;\mathbb{C}})$ in a regular domain Ω of the complex plane then

$$\frac{1}{i}\int_{\partial\Omega} w(z)\,dz = \int_{\Omega} D_{\overline{z}}w(z)\,dxdy \tag{7}$$

where $D_{\overline{z}}$ is twice the Cauchy-Riemann operator.

Theorem 2.11. **Cauchy formula** Let $\Omega' \subset \mathbb{C}$ be a regular domain and $h \in C^1(\Omega';\mathbb{C}) \cap C(\overline{\Omega'};\mathbb{C})$. Then, using $\zeta = \xi + i\eta$ for $z \in \Omega'$,

$$h(z) = \frac{1}{i}\int_{\partial\Omega'} K(\zeta - z)h(\zeta)\,d\zeta. \tag{8}$$

where $K(z) = \frac{1}{2\pi z}$ is the Cauchy's kernel

Proof 3. See Begehr (2005); Ablowitz and Fokas (2003) for the proof.

Theorem 2.12. Let f be analytic in a domain Ω and continuously differentiable with respect to x and y in the closure $\overline{\Omega}$. Then $f(\zeta)$ can be written locally in the form

$$f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - z_0)^n, \tag{9}$$

where

$$a_n = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}}dz. \tag{10}$$

Proof 4. See Tutschke and Vasudeva (2004) for the proof.

Theorem 2.13. Let w be a complex-valued function such that $w \in C^1(\Omega,\mathbb{C})$ that satisfies

$$\partial_{\overline{z}}(-w) = g = A + B\frac{\overline{w}}{w}. \tag{11}$$

Then w is generalized analytic if and only if $\Phi = we^{-w}$ is analytic.

Proof 5. See Vekua (1962) for the proof.

2.3 Fractional complex calculus

The fractional Cauchy–Riemann operator is defined on a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ such that $f \in AC(\Omega,\mathbb{C})$ by

$$D^{\alpha}_{\overline{z}}[f] = D^{\alpha_0}_{x}[f] + iD^{\alpha_1}_{y}[f],$$

where $\alpha \in (0,1)$.

Definition 2.14. A complex-valued function $f \in AC(\Omega,\mathbb{C})$ is called *fractional analytic* over Ω, if it satisfies the fractional Cauchy-Riemann equation given by $D^{\alpha}_{\overline{z}}[f] = 0$.

Theorem 2.15. Let f be a complex valued function, such that $f = f_0 + if_1$, where $f_0, f_1 \in AC^2(\Omega)$, $x = x_0$, $y = x_1$ and

$$f_0(x_0^+, x_1) = f_1(x_0, x_1^+) = 0.$$

Then the following statements are true:

(i) $D^{\alpha_k}_{x_k}(D^{\alpha_k}_{x_k} f_j) = D^{2\alpha_k}_{x_k} f_j$ for $k, j \in \{0,1\}$.
(ii) $D^{\alpha_l}_{x_l}(D^{\alpha_k}_{x_k} f_j) = D^{\alpha_k}_{x_k}(D^{\alpha_l}_{x_l} f_j)$ for $k \neq j$.

Proof 6. The proof follows by direct calculation, using the results presented in Ceballos et al., Theorem. 3.2. Ceballos et al. (2020).

3. QUASI-FRACTIONAL ANALYTIC FUNCTIONS

Definition 3.1. A function $f \in AC(\Omega,\mathbb{C})$ such that $f = u + iv$ is quasi-fractional analytic on Ω if it satisfies the following system of equations:

$$\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v = 0, \tag{12}$$

$$D_x^{\alpha}v + D_y^{\alpha}u = 0. \tag{13}$$

Example 3.2. Let $\alpha \in (0,1)$ and define $f \in AC(\Omega,\mathbb{C})$ by

$$f(x,y) = (x - a_0)(y - a_1)^{\alpha-1} + i(x - a_0)^{\alpha-1}(y - a_1).$$

Using Example 2.7, we get

$$I_y^{1-\alpha}u = (x - a_0)C_{\alpha}$$

where $C_{\alpha} = \frac{\pi\csc(\pi\lambda)}{\Gamma(1-\lambda)} \in \mathbb{R}$. Thus,

$$\partial_x I_y^{1-\alpha}u = C_{\alpha}.$$

Similarly,

$$I_x^{1-\alpha}v = (y - a_1)C_{\alpha},$$
$$\partial_y I_x^{1-\alpha}v = C_{\alpha}.$$

Thus,

$$\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v = C_{\alpha} - C_{\alpha} = 0,$$

and equation (12) is satisfied.

On the other hand, by Example 2.6 it follows that:

$$D_x^{\alpha}v = (y - a_1)D_x^{\alpha}[(x - a_0)^{\alpha-1}] = 0,$$
$$D_y^{\alpha}u = (x - a_0)D_y^{\alpha}[(y - a_1)^{\alpha-1}] = 0,$$

satisfying (13). Therefore, f is quasi-fractional analytic. Note that f is not analytic.

Lemma 3.3. A complex-valued function $f = u + iv$ in $AC(\Omega,\mathbb{C})$ is quasi-fractional analytic if and only if $g = I_y^{1-\alpha}u + iI_x^{1-\alpha}v$ satisfies the Cauchy-Riemann equation on Ω.

Proof 7. Necessity. Suppose $f = u + iv$ is absolutely continuous in Ω. Then the function $g = I_y^{1-\alpha}u + iI_x^{1-\alpha}v$ exists (is defined) in Ω and $g \in AC^1(\Omega,\mathbb{C})$. Applying the Cauchy-Riemann operator over g, we get

$$D_{\overline{z}}g(z) = (\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v) + i(D_x^{\alpha}v + D_y^{\alpha}u).$$

By hypothesis, real and imaginary part of $D_{\overline{z}}g(z)$ are both zero, so $D_{\overline{z}}g(z) = 0$, i.e. g is analytic in Ω.

Sufficiency. Let $f = u + iv$ be a complex-valued function defined on $\Omega \subset \mathbb{C}$. Suppose the function $g = I_y^{1-\alpha}u + iI_x^{1-\alpha}v$ is defined and analytic on Ω, i.e. $D_{\overline{z}}g(z) = 0$, then

$$\begin{aligned} D_{\overline{z}}g(z) &= (\partial_x + i\partial_y)(I_y^{1-\alpha}u + iI_x^{1-\alpha}v) \\ &= (\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v) + i(\partial_x I_x^{1-\alpha}v + \partial_y I_y^{1-\alpha}u) \\ &= (\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v) + i(D_x^{\alpha}v + D_y^{\alpha}u) \\ &= 0 \end{aligned}$$

which leads to the following system of equations

$$\partial_x I_y^{1-\alpha}u - \partial_y I_x^{1-\alpha}v = 0, \tag{14}$$

$$D_x^{\alpha}v + D_y^{\alpha}u = 0. \tag{15}$$

Hence, f is quasi-fractional analytic.

Definition 3.4. A function $f \in AC^1(\Omega,\mathbb{C})$ such that $f = u + iv$ is anti-quasi-fractional analytic on Ω if it satisfies the following system of equations:

$$\partial_x I_y^{1-\alpha}u + \partial_y I_x^{1-\alpha}v = 0, \tag{16}$$

$$D_x^{\alpha}v - D_y^{\alpha}u = 0. \tag{17}$$

Example 3.5. Let $\alpha \in (0,1)$ and let $f \in AC^1(\Omega,\mathbb{C})$ be defined by

$$f(x,y) = (x - a_0)(y - a_1)^{\alpha-1} - i(x - a_0)^{\alpha-1}(y - a_1).$$

Then f is anti-quasi-fractional analytic in Ω, by calculations similar to those in Example 3.2.

The link between f and g is given by the following expressions

$$g(z) = I_y^{1-\alpha} Re[f(z)] + i I_x^{1-\alpha} Im[f(z)],$$
$$f(z) = D_y^{1-\alpha} Re[g(z)] + i D_x^{1-\alpha} Im[g(z)].$$

The second expression follows from the fact that

$$D_x^{1-\alpha} I_x^{1-\alpha} h(z) = h(z)$$

for any $h \in AC(\Omega, \mathbb{R})$. Using the well-known identities $Re[f] = \frac{f+\overline{f}}{2}$ and $Im[f] = \frac{f-\overline{f}}{2i}$ in the expression above, we get

$$g(z) = \frac{1}{2}[(I_y^{1-\alpha} + I_x^{1-\alpha}) f(z) + (I_y^{1-\alpha} - I_x^{1-\alpha})\overline{f(z)}].$$

In a similar way, we can get an expression for $f(z)$ in terms of $g(z)$ and $\overline{g(z)}$,

$$f(z) = \frac{1}{2}[(D_y^{1-\alpha} + D_x^{1-\alpha}) g(z) + (D_y^{1-\alpha} - D_x^{1-\alpha})\overline{g(z)}]. \tag{18}$$

Lemma 3.6. Let $\Omega \subset \mathbb{C}$ be a regular domain and $f = u + iv$ quasi-fractional analytic such that $f \in C^1(\Omega'; \mathbb{C}) \cap C(\overline{\Omega}; \mathbb{C})$. Then, using $\zeta = \xi + i\eta$

$$f(z) = \frac{1}{2i} \int_{\partial\Omega} [\mathfrak{K}_\alpha(\zeta, z) g(\zeta)\, d\zeta - \mathfrak{L}_\alpha(\overline{\zeta}, \overline{z}) \overline{g(\zeta)}\, d\overline{\zeta}] \tag{19}$$

where

$$\mathfrak{K}_\alpha(\zeta, z) = (D_y^{1-\alpha} + D_x^{1-\alpha}) K(\zeta - z)$$

and

$$\mathfrak{L}_\alpha(\zeta, z) = (D_y^{1-\alpha} - D_x^{1-\alpha}) K(\zeta - z)$$

Proof 8. Suppose f is quasi-fractional analytic, so g is analytic. Moreover, g has the following integral representation

$$g(z) = \frac{1}{i} \int_{\partial\Omega} K(\zeta - z) g(\zeta)\, d\zeta,$$

where $K(z) = \frac{1}{2\pi z}$ is the Cauchy's kernel. Using this formula on (18), we get the desired expression

$$f(z) = \frac{1}{2i} \int_{\partial\Omega} [\mathfrak{K}_\alpha(\zeta, z) g(\zeta)\, d\zeta - \mathfrak{L}_\alpha(\overline{\zeta}, \overline{z}) \overline{g(\zeta)}\, d\overline{\zeta}]. \tag{20}$$

where

$$\mathfrak{K}_\alpha(\zeta, z) = (D_y^{1-\alpha} + D_x^{1-\alpha}) K(\zeta - z)$$

and

$$\mathfrak{L}_\alpha(\zeta, z) = (D_y^{1-\alpha} - D_x^{1-\alpha}) K(\zeta - z)$$

Note: $\overline{K(\zeta - z)} = K(\overline{\zeta} - \overline{z})$.

Lemma 3.7. Let $\Omega \subset \mathbb{C}$ be a regular domain and $f = u + iv \in AC(\Omega, \mathbb{C})$ be quasi-fractional analytic on Ω. Suppose that the fractional integrals $I_y^{1-\alpha} u$ and $I_x^{1-\alpha} v$ are well-defined and continuously differentiable with respect to x and y in $\overline{\Omega}$. Then

$$I_y^{1-\alpha} u + i I_x^{1-\alpha} v = \sum_{n=0}^{\infty} a_n (\zeta - z_0)^n \tag{21}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{I_y^{1-\alpha} u + i I_x^{1-\alpha} v}{(z - z_0)^{n+1}}\, dz. \tag{22}$$

Proof 9. Since f is quasi-fractional analytic, by lemma 3.3 we know that

$$g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$$

is analytic on Ω. Since g is also continuously differentiable on $\overline{\Omega}$, by Theorem 2.12, the power series representation (21) follows immediately.

Lemma 3.8. Let $0 < \alpha < 1$, $\gamma \neq 0$ be constants, and let $f(t)$ be a function in $AC(a, b)$ with $a, b \in \mathbb{R}$ such that $I_t^\alpha f(t)$ exists. If the following holds:

$$I_t^\alpha f(t) = \gamma f(t).$$

then $f(t) = 0$ for all $t \in (a, b)$.

Proof 10. According to Samko et al. (1993), the solution to the Abel integral equation

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\zeta) d\zeta}{(t - \zeta)^{1-\alpha}} = f(t) \tag{23}$$

is uniquely given by

$$\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\zeta)}{(t - \zeta)^\alpha}\, d\zeta. \tag{24}$$

Substituting $\varphi(t) = \gamma^{-1} f(t)$ into (23) and (24), we obtain:

$$I_t^\alpha f(t) = \gamma f(t)$$
$$D_t^\alpha f(t) = \gamma f(t)$$

Which implies that $f(t) = 0$.

Theorem 3.9. Let Ω be a domain containing the origin, and let $f = u + iv$ be analytic and non-constant on Ω, such that u and v are both non-constant separable functions given by:

$$u = h_1(x) f_1(y), \quad v = h_2(x) f_2(y)$$

where each function composing u and v belongs to $AC(\Omega)$. Then f is not quasi-fractional.

Proof 11. We proceed by contradiction. Suppose that f is quasi-fractional. Then, for every $z = x + iy \in \Omega$, the following equations hold:

$$\frac{\partial}{\partial x} h_1(x) I_y^{1-\alpha} f_1(y) - \frac{\partial}{\partial y} f_2(y) I_x^{1-\alpha} h_2(x) = 0, \tag{25}$$
$$f_2(y) D_x^\alpha h_2(x) + h_1(x) D_y^\alpha f_1(y) = 0. \tag{26}$$

Since f is analytic, the Cauchy–Riemann equations also hold:

$$\frac{\partial}{\partial x} h_1(x) f_1(y) - \frac{\partial}{\partial y} f_2(y) h_2(x) = 0, \tag{27}$$
$$\frac{\partial}{\partial y} f_1(y) h_1(x) + \frac{\partial}{\partial x} h_2(x) f_2(y) = 0. \tag{28}$$

Applying the operator $I_y^{1-\alpha}$ to (27), we obtain:

$$\frac{\partial}{\partial x} h_1(x) I_y^{1-\alpha} f_1(y) - I_y^{1-\alpha} \left(\frac{\partial}{\partial y} f_2(y) \right) h_2(x) = 0. \tag{29}$$

Subtracting (29) from (25) yields:

$$\frac{\partial}{\partial y} f_2(y) I_x^{1-\alpha} h_2(x) = h_2(x) I_y^{1-\alpha} \left(\frac{\partial}{\partial y} f_2(y) \right). \tag{28}$$

Now consider those points $z \in \Omega$ for which both $\frac{\partial}{\partial y} f_2(y) \neq 0$ and $h_2(x) \neq 0$. Then:

$$\frac{I_y^{1-\alpha}(\frac{\partial}{\partial y} f_2(y))}{\frac{\partial}{\partial y} f_2(y)} = \frac{I_y^{1-\alpha} h_2(x)}{h_2(x)}.$$

By the fact that each side of the previous equation depends on a different variable, this implies that the ratio of the two functions must be a real constant. However, using Lemma 3.8, we find that this can only occur if $h_2(x) = 0$

or $\frac{\partial}{\partial y} f_2(y) = 0$. By hypothesis, $u \neq 0$, so $\frac{\partial}{\partial y} f_2(y) = 0$, which implies $f_2(y) = c_1$, where $c_1 \neq 0$ is a real constant.

Substituting this into (27), we obtain:

$$\frac{\partial}{\partial x} h_1(x) f_1(y) = 0.$$

Since $v \neq 0$, it follows that $\frac{\partial}{\partial x} h_1(x) = 0$, i.e., $h_1(x) = c_2$ with $c_2 \neq 0$ constant. Substituting into (28), we find:

$$c_2 \frac{\partial}{\partial y} f_1(y) + c_1 \frac{\partial}{\partial x} h_2(x) = 0. \tag{30}$$

Suppose now that $f_1(y)$ and $h_2(x)$ are not constant functions satisfying the previous relation. Applying the operator $I_y^{1-\alpha}$ to (30), we get:

$$c_2 I_y^{1-\alpha}\left(\tfrac{\partial}{\partial y} f_1(y)\right) + c_1 \frac{\partial}{\partial x} h_2(x) I_y^{1-\alpha}(1) = 0. \tag{31}$$

Noting that:

$$I_y^{1-\alpha}\left(\frac{\partial}{\partial y} f_1(y)\right) = \frac{\partial}{\partial y}\left(I_y^{1-\alpha} f_1\right) - \frac{f_1(0) y^{-\alpha}}{\Gamma(\alpha)}. \tag{32}$$

Substituting this into the previous expression, we obtain

$$c_2 \left(\frac{\partial}{\partial y}\left(I_y^{1-\alpha} f_1\right) - \frac{f_1(0) y^{-\alpha}}{\Gamma(\alpha)}\right) + c_1 \frac{\partial}{\partial x} h_2(x) I_y^{1-\alpha}(1) = 0. \tag{33}$$

Thus

$$\frac{\partial}{\partial x} h_2(x) I_y^{1-\alpha}(1) = \frac{c_2}{c_1}\left(\frac{f_1(0) y^{-\alpha}}{\Gamma(\alpha)} - D_y^{\alpha} f_1(y)\right) \tag{34}$$

Since the right-hand side of (34) depends only on y, this implies that $\frac{\partial h_2(x)}{\partial x}$ must be constant. However, if it is constant, then the equality only holds for a specific value of y different from cero. Therefore, (30) is only satisfied when both $\frac{\partial}{\partial y} f_1(y) = 0$ and $\frac{\partial}{\partial x} h_2(x) = 0$, which implies that f is constant a contradiction.

3.1 The connection between harmonicity and analyticity using fractional calculus

In this section, we explore the relationship between harmonicity and analyticity for a function g defined via Riemann Liouville integrals. The approach is to begin with harmonicity and, through the tools of fractional calculus, characterize a particular class of analytic functions.

Lemma 3.10. If $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$ is harmonic and $\partial_{\overline{z}} g = \Psi$, then Ψ is anti-analytic.

Proof 12. Let g be harmonic, that is, $\partial_z \partial_{\overline{z}} g = 0$. Consider

$$\partial_{\overline{z}} g = (\partial_x + i\partial_y)(I_y^{1-\alpha} u + i I_x^{1-\alpha} v) =$$
$$\partial_x I_y^{1-\alpha} u - i\partial_y I_x^{1-\alpha} v + i[D_x^{\alpha} v + D_y^{\alpha} u]. \tag{35}$$

Let

$$\varphi_1 = \partial_x I_y^{1-\alpha} u - \partial_y I_x^{1-\alpha} v, \tag{36}$$

$$\varphi_2 = D_x^{\alpha} v + D_y^{\alpha} u. \tag{37}$$

Accordingly, we express $\Psi = \varphi_1 + i\varphi_2$. By applying ∂_z to equation (35), and using that g is harmonic, we obtain:

$$(\partial_x - i\partial_y)\Psi = (\partial_x - i\partial_y)(\partial_{\overline{z}} g) = 0. \tag{38}$$

Lemma 3.11. If $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$ is harmonic and $f = u(y) + iv(x)$ then $D_x^{\alpha} v + D_y^{\alpha} u$ is analytic.

Proof 13. Let g be harmonic. Then,

$$\Delta g = \partial_x[\partial_x I_y^{1-\alpha} u - \partial_y I_x^{1-\alpha} v] + \partial_y[D_x^{\alpha} v + D_y^{\alpha} u]$$
$$-i(\partial_y[\partial_x I_y^{1-\alpha} u - \partial_y I_x^{1-\alpha} v] - \partial_x[D_x^{\alpha} v + D_y^{\alpha} u]) = 0. \tag{39}$$

Since $u = u(y)$ and $v = v(x)$, it follows that $D_x^{\alpha} v + D_y^{\alpha}$ is analytic

Lemma 3.12.

- If $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$ is harmonic and $f = u(x,y) + iv(x,y)$ and $\partial_x I_y^{1-\alpha} u - \partial_y I_x^{1-\alpha} v = 0$ then $D_x^{\alpha} v + D_y^{\alpha} u$ is analytic.
- If $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$ is harmonic and $f = u(x,y) + iv(x,y)$ and $D_x^{\alpha} v + D_y^{\alpha} u = 0$ then $\partial_x I_y^{1-\alpha} u - \partial_y I_x^{1-\alpha} v$ is analytic.

Proof 14. Both proofs follow by direct calculation using system (39) with $f = u(x,y) + iv(x,y)$.

4. QUASI-GENERALIZED FRACTIONAL ANALYTIC FUNCTIONS

Theorem 4.1. Let $f = u + iv$ be a complex-valued function. Suppose $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$ is a generalized-analytic function, i.e. g satisfies

$$g_{\overline{z}} + Ag + B\overline{g} = 0.$$

where A and B are complex valued functions and $-g_{\overline{z}} = A + B\frac{\overline{g}}{g}$. Then, f is quasi-fractional analytic.

Proof 15. Since $-g_{\overline{z}} g = Ag + B\overline{g}$, it follows that:

$$g_{\overline{z}} - g_{\overline{z}} g = 0$$

or equivalently,

$$g_{\overline{z}}(1 - g) = 0.$$

Then if $g_{\overline{z}} = 0$, f is quasi-fractional analytic. If that is not the case, then:

$$g = 1,$$
$$I_y^{1-\alpha} u + i I_x^{1-\alpha} v = 1.$$

And applying the Cauchy-Riemann operator gives:

$$(\partial_x + i\partial_y)(I_y^{1-\alpha} u + i I_x^{1-\alpha} v) = 0.$$

Which again shows that f is necessarily quasi-fractional analytic in either case.

Lemma 4.2. Let $f = u + iv$ be a quasi-fractional analytic function on Ω such that $g = I_y^{1-\alpha} u + i I_x^{1-\alpha} v$. Then $w = (I_y^{1-\alpha} u + i I_x^{1-\alpha} v)e^{-g}$ is a generalized analytic function.

Proof 16. Since f is quasi-fractional analytic, it follows from Definition 3.1 that g is analytic. By Theorem 2.13, the analyticity of g implies that the function w, is generalized analytic. This can be expressed by the following diagram:

$$\begin{tikzcd}
\text{Quasi-Fractional Analytic} \arrow[r] \arrow[dr] & \text{Analytic} \arrow[d] \\
& \text{Generalized Analytic}
\end{tikzcd}$$

5. CONCLUSION

This work presents a study of quasi-fractional analytic functions, a new class of functions that combines concepts from complex analysis and fractional calculus. By extending the classical Cauchy-Riemann framework to include Riemann–Liouville operators, we established a system of conditions under which a function is considered quasi-fractional analytic.

A key result is the equivalence between the quasi-fractional analyticity of a function and the classical analyticity of an auxiliary function constructed from fractional integrals. This connection enables the application of classical complex analysis results, including power series expansions and integral representations.

We further explored the relationship between harmonicity and quasi-fractional analyticity, showing how fractional calculus tools can characterize families of analytic and generalized analytic functions. Additionally, the paper identified limitations by proving that classical analytic functions with separable real and imaginary parts cannot be quasi-fractional analytic.

The results presented here open the path for future work in fractional complex analysis and its applications in hypercomplex algebras, physics and engineering problems where classical tools are insufficient.

REFERENCES

Ablowitz, M. and Fokas, A. (2003). *Complex variables: introduction and applications*. Cambridge University Press.

Alsaedi, A., Nieto, J., and Venktesh, V. (2015). Fractional electrical circuits. *Advances in Mechanical Engineering*. doi:10.1177/1687814015618127.

Begehr, H. (1996). Complex analytic methods for partial differential equations. *Zeitschrift für angewandte Mathematik und Mechanik*, 76, 21–24.

Begehr, H. (2005). Boundary value problems in complex analysis i. *Bol. Asoc. Mat. Venez*, 12(1), 65–85.

Ceballos, J., Coloma, N., Di Teodoro, A., and Ochoa-Tocachi, D. (2020). Generalized fractional Cauchy–Riemann operator associated with the fractional cauchy-riemann operator. *Advances in Applied Clifford Algebras*, 30, 1–22.

Ceballos, J., Coloma, N., Di Teodoro, A., Ochoa-Tocachi, D., and Ponce, F. (2022). Fractional multicomplex polynomials. *Complex Analysis and Operator Theory*, 16(4), 60.

Coloma, N., Di Teodoro, A., Ochoa-Tocachi, D., and Ponce, F. (2021). Fractional elementary bicomplex functions in the riemann-liouville sense. *Advances in Applied Clifford Algebras*, 31(4), 63.

Diethelm, K. (2010). *The Analysis of Fractional Differential Equations, An Application-Oriented, Exposition Using Differential Operators of Caputo type*, volume 2004 of *Lecture Notes in Mathematics*. Springer, Heidelbereg.

González-Cervantes, J. and Bory-Reyes, J. (2024). A bicomplex (ϑ, φ)-weighted fractional Borel-Pompeiu type formula. *Differential Equations*.

Hilfer, R. (2003). *Applications of Fractional Calculus in Physics*. World Scientific Publishing Co, Singapore.

Kähler, U. and Vieira, N. (2014). Fractional clifford analysis. In S. Bernstein, U. Kähler, I. Sabadini, and F. Sommen (eds.), *Hypercomplex Analysis: New Perspectives and Applications*, Trends in Mathematics, 191–201. Birkhäuser, Basel.

Kilbas, A. and Marzan, S. (2005). Non linear differential equations with Caputo fractional derivative in the space of continuously differentiable functions. *Differ. Equ.*, 41, 84–95.

Kilbas, A., Srivastava, H., and Trujillo, J. (2006). *Theory and Applications of Fractional Differential Equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier, Amsterdam.

Lovoie, J., Osler, T., and Tremblay, R. (1976). Fractional derivatives and special functions. *SIAM Rev.*, 18(2), 240–268.

Magin, R. (2006). *Fractional Calculus in Bioengineering*. Begell House Inc. Publishers.

Miller, K. and Ross, B. (1993). *An Introduction to Fractional Calculus and Fractional Differential Equations*. Wiley, New York.

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego.

Sabatier, J., Agrawal, O.P., and Machado, J. (2007). *Advances in fractional calculus*, volume 4. Springer.

Samko, S., Kilbas, A., and Marichev, O. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon.

Struppa, D., Vajiac, A., and Vajiac, M. (2012). Holomorphy in multicomplex spaces. In *Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations*, volume 221 of *Operator Theory: Advances and Applications*, 617–634. Birkhäuser Basel.

Torres, D. and Malinowska, A. (2012). *Introduction to the Fractional Calculus of Variations*. Imperial College Press.

Tutschke, W. and Vasudeva, H.L. (2004). *An Introduction to Complex Analysis: Classical and Modern Approaches*. CRC Press.

Vekua, I.N. (1962). *Generalized Analytic Functions*. Pergamon.