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IFAC PapersOnLine 59-37 (2025) 47–52

An Introduction to Weighted Operators via Composition and Selected Properties, Aimed at Numerical Implementation ⋆

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Abstract: This work introduces a novel class of weighted fractional operators constructed through the composition of differential and integral operators. In particular, we propose the operator ${}^{\overline{q}}D^{\mu}_{x}$, which generalizes classical fractional derivatives while maintaining essential properties such as linearity. Although the semigroup property and the Leibniz rule do not hold in their traditional forms, we derive analogous formulations by combining the proposed operator with the Riemann–Liouville derivative. Furthermore, a numerical representation based on the Grünwald–Letnikov method is developed, enabling efficient discretization and simulation of the weighted operator in cases where analytical solutions are intractable. The approach also considers the interplay between Laplace transforms and convolutions, which is crucial for real-world applications in control and signal processing.

Keywords: Weighted Operators, Composition properties, Fractional systems, Fractional PID control, Industrial Process Control.

1. INTRODUCTION

Fractional calculus, the study of differentiation and integration to non-integer orders, has evolved significantly since its beginning, with numerous operators proposed in (Caputo and Fabrizio (2016, 2015); Bouzeffour and Jedidi (2024)) to model complex phenomena in physics, engineering, and applied mathematics (Baleanu et al. (2010); Magin (2006); Naber (2004)). While the Riemann-Liouville and Caputo fractional derivatives have served as fundamental building blocks, recent research (Fernandez and Fahad (2022)) has focused on generalized operators that extend this theory to better suit specific applications.

One such area of application is control theory. The Fractional Order PID (FOPID) controller, commonly expressed as $PI^{\lambda}D^{\mu}$, extends the classical PID framework by introducing two additional parameters, λ and μ, which correspond to the integral and derivative orders, respectively. These fractional exponents offer enhanced tuning capability, allowing for finer adjustments of both time-domain and frequency-domain behavior. This added flexibility makes FOPID controllers particularly effective for systems with complex dynamics, such as those involving memory effects, diffusion phenomena, or viscoelastic behavior—scenarios where traditional integer-order controllers may not perform adequately (Yokuş et al. (2024)). However, to ensure optimal performance under varying operating conditions, further adaptations and refinements of the controller structure are often necessary (Gude et al. (2024a)).

In this context, weighted fractional operators have emerged as particularly valuable, due to their ability to incorporate adjustable weights that modify the differentiation and integration processes (Padula and Visioli (2013); Bingi et al. (2018)).

This work studies a new class of weighted fractional operators constructed via composition, where differential and integral operators are combined to define new behaviors. Specifically, we introduce a weighted operator ${}^{\overline{q}}D^{\mu}_{x}$, which generalizes traditional fractional derivatives while preserving key properties such as linearity. Although the semigroup property and the Leibniz rule do not generally hold, we propose analogous properties by combining the weighted operator with the Riemann–Liouville derivative.

Moreover, this work includes the numerical representation of the q-weighted operator using the Grünwald–Letnikov approach, which provides a direct discretization of fractional derivatives, thereby enabling computational simulations in scenarios where analytical solutions are intractable. This approach also considers the interplay between Laplace transforms and convolutions across different domains, which is essential for practical applications.

The structure of this paper is as follows. In Section 2, we recall fundamental concepts in fractional calculus, including the definitions of the Riemann-Liouville and Caputo

⋆ This work was supported by the Basque Government (BEREZ-IA Elkartek Project, Grant KK-2023/00012) and by Universidad San Francisco de Quito (USFQ) through the Poli-Grants Program (Grant 33603).

10.1016/j.ifacol.2026.01.009

fractional operators, as well as their key properties. Additionally, we review the theory of operator composition methods that form the basis for our weighted operator construction. Sections 3 and 4 develop our main contributions, introducing the weighted fractional operator and establishing its fundamental properties. We prove essential operator identities and composition rules, while demonstrating how classical fractional operators emerge as special cases. Section 5 presents the numerical implementation of our framework using Grünwald-Letnikov approximation methods. Finally, we compare our results with existing approaches in the literature and discuss potential applications in control theory and differential equations.

2. FRACTIONAL CALCULUS

Definition 1. Let $\left(D_a^\mu f\right)(t)$ denote the **fractional Riemann Liouville derivative** of order $\mu > 0$ (see Kilbas et al. (2006); Podlubny (1999a); Samko et al. (1993)) and let it be defined as:

$$\left(D_a^\mu f\right)(t) = \left(\frac{d}{dt}\right)^p \left(I_a^{s-\mu} f\right)(t) = \left(\frac{d}{dt}\right)^p \frac{1}{\Gamma(p-\mu)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-p-\mu}}\, d\tau, \tag{1}$$

where $p = [\mu] + 1$, $t > a$, $[\mu]$ denotes the integer part of μ and Γ is the gamma function.

Remark 1. The function $f \in L_1(a,b)$ is said to have a summable fractional derivative $\left(D_a^\mu f\right)(t)$ if $\left(I_a^{p-\mu} f\right)(t) \in AC^p([a,b])$, where $p = [\mu]+1$ and $AC^p([a,b])$ denotes the class of functions f whose $p-1$-th derivative is absolutely continuous on $[a,b]$.

When $0 < \mu < 1$, expression (1) simplifies to: $\left(D_a^\mu f\right)(t) = \frac{d}{dt}\left(I_a^{1-\mu} f\right)(t)$.

Observe that as $\mu \to 1$, the standard derivative operator is recovered (Kilbas et al. (2006); Podlubny (1999a); Samko et al. (1993)). The semigroup property for the composition of fractional derivatives does not hold generally (see (Podlubny, 1999a, Sect. 2.3.6)). In fact, the property:

$$D_a^\mu \left(D_a^\beta h\right) = D_a^{\mu+\beta} h, \tag{2}$$

holds if

$$h^{(j)}(a) = 0, \qquad j = 0, 1, \ldots, s-1, \tag{3}$$

and $h \in AC^{s-1}([a,b])$, $h^{(s)} \in L_1(a,b)$ and $s = [\beta]+1$. Similarly, the Leibniz rule doesn't hold generally either (See Samko et al. (1993), Cap. 15) but the fractional derivative of a product can take the form

$$D_a^\mu[f(x)g(x)] = \sum_{k=0}^{\infty} \binom{\mu}{k} f^{(k)}(x)(D_a^{\mu-k} g)(x). \tag{4}$$

If the real-valued functions f, g are analytic on $[a,b]$, where

$$\binom{\mu}{k} = \frac{\Gamma(\mu+1)}{k!\,\Gamma(\mu-k+1)}. \tag{5}$$

Example 1. Let $\mu, \lambda \in$ (0,1), $a > 0$, k $\in \mathbb{N}$ and $\beta > -1$, then

(1) $I_x^\lambda\left[(x-a)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\lambda+1)}(x-a)^{\beta+\lambda}$.

(2) $D_a^\mu\left[(x-a)^\beta\right] = \begin{cases} 0, \ \beta = \mu - 1, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\mu+1)}(x-a)^{\beta-\mu}, \text{ otherwise.} \end{cases}$

Remark 1. It is worth noticing that the **Riemann-Liouville derivative of a constant is not zero**. However, in the limit process, it behaves as expected.

$$\lim_{\mu\to 1} \left(D_a^\mu 1\right)(x) = \lim_{\mu \to 1} \frac{(x-a)^{-\mu}}{\Gamma(1-\mu)} = 0. \tag{6}$$

3. FRACTIONAL COMPOSITIONS OPERATORS

The main motivation of this section is to show how fractional operators can be viewed as various compositions of differential and integral-type operators. Based on this idea, we decided to experiment with different operators, which eventually led us to the central operator of this paper.

Let Θ_d be a differential operator and Θ_i, an integral operator. $Comp = \Theta_d \circ \Theta_i \circ \cdots \circ \Theta_d \circ \Theta_i$

For example, consider the following composition of three operators, where $\mu \in (0,1)$, Θ_d is the ordinary differential operator and Θ_i is the fractional integral of order μ

$$Comp_{[\mu]} = \frac{d}{dt} \circ I_a^{1-\mu} \circ \frac{d}{dx} = \frac{d}{dt}\left(I_a^{1-\mu}\frac{d}{dx}\right).$$

This can be interpreted either as the derivative of a Caputo derivative, or the Riemann-Liouville derivative of a derivative.

As another possibility, consider $\mu \in (0,1)$, m, n can be 1 or 0, and define $\frac{d^0}{dx^0} f(t) = f(t)$. Then $Comp_{[n,m,\mu]}(f)[t] = \dfrac{d^n}{dt^n} \circ I_a^{1-\mu} \circ \dfrac{d^m}{dx^m} f(x)$ If $n = 1$ and $m = 0$ we obtain

$$\begin{aligned} Comp_{[n=1,m=0,\mu]}(f)[t] &= \frac{d}{dt} \circ I_a^{1-\mu} \circ \frac{d^0}{dx^0} f(x) \\ &= \frac{d}{dt} I_a^{1-\mu} f(x) = D_a^\mu f. \end{aligned} \tag{7}$$

That is, the Riemann-Liouville derivative of order μ. Alternatively, if $n = 0$ and $m = 1$, we obtain

$$Comp_{[n=0,m=1,\mu]}(f)[t] = \frac{d^0}{dt^0} \circ I_a^{1-\mu} \circ \frac{d}{dx} f(x) \tag{8}$$

$$= I_a^{1-\mu} \frac{d}{dx} f(x) = {}_cD_{t+}^\mu f. \tag{9}$$

Which is the Caputo derivative of order μ.
Now, consider $\Theta_d = \frac{d^n}{dt^n}$, and denote the integral operator on the right as $I_a^{(1-\alpha)(1-\beta)}$, and the integral operator on the left as $I_a^{\beta(1-\alpha)}$. Then, we can write

$$Comp_{[n=1,m=0,\alpha]}(f)[t] = I_a^{\beta(1-\alpha)} \frac{d^n}{dt^n} I_a^{(1-\alpha)(1-\beta)} f. \tag{10}$$

This composition defines a more general fractional operator known as Hilfer fractional derivative (see Tomovski et al. (2010)).
Hence, depending on the choice of parameters, the *Comp* operator can represent the Hilfer, Riemann-Liouville or Caputo Operator (see Samko et al. (1993)). However, this operator can be generalized further. For instance, consider the composition of four operators, where $\alpha, \beta \in (0,1)$

$$Comp_{[n=1,m=0,\alpha,\beta]}(f)[t] = \frac{d^n}{dt^n} \circ I_a^{n-\alpha} \circ \frac{d^m}{dw^m} \circ I_a^{m-\beta} f(w). = D_a^{\alpha} g(t).$$

Where $g(t) = D_a^{\beta} f(w)$.

Remark 2. In the last example, we considered that $I_x^{-\beta} f(x) = \partial_x^{\beta} f(x)$. (see Samko et al. (1993)).

Definition 2. Consider $q_1(t,\mu), \gamma_1(t,\mu)$ a continuous function, $q_2(t,\mu), \gamma_2(t,\mu)$ continuously differentiable functions on $t > a$ and let $\left({}^{\overline{q},\overline{\gamma}}D_a^{\mu} f\right)(t) = \left({}^{(q_1,q_2),(\gamma_1,\gamma_2)}D_a^{\mu} f\right)(t)$ denote the **$\overline{q}$-weighted generalized fractional Riemann-Liouville derivative** of order $\mu > 0$. For $q_1, q_2, \gamma_1\gamma_2 \in AC^s(\mathbb{R})$

$$\left({}^{\overline{q},\overline{\gamma}}D_a^{\mu} f\right)(t) = \left(q_1(t,\mu)\frac{d}{dt} - \gamma_1\right)^s \left(q_2(t,\mu)\left(I_a^{s-\mu} f\right) - \gamma_2\right). \quad (11)$$

where $s = [\mu] + 1, t > a$ and $[\mu]$ denotes the integer part of μ.

Definition 3. Consider $q_1(t,\mu), q_2(t,\mu), \gamma_1(t,\mu), \gamma_2(t,\mu) \in AC^s(\mathbb{R})$ and let $\left({}^{\overline{q},\overline{\gamma}}\mathcal{D}_a^{\mu} f\right)(t) = \left({}^{(q_1,q_2),(\gamma_1,\gamma_2)}\mathcal{D}_a^{\mu} f\right)(t)$ denote the **$\overline{q}$-weighted generalized fractional Caputo derivative** of order $\mu > 0$.

$$\left({}^{\overline{q},\overline{\gamma}}\mathcal{D}_a^{\mu} f\right)(t) = \left(q_1(t,\mu)I_{a^+}^{s-\mu} - \gamma_1\right)^s \left(q_2(t,\mu)\frac{d}{dt}f - \gamma_2\right). \quad (12)$$

where $s = [\mu] + 1, t > a$ and $[\mu]$ denotes the integer part of μ.

Based on this idea of composition, we can motivate the weighted operators inspired by papers Fernandez and Fahad (2022); Katugampola (2011); Sousa and De Oliveira (2018). The use of "weights" allows for an even more general approach to the composition of integral and differential operators. Let γ_i (cte), take q_i as functions of $(t,\mu)$and n_i as non-negative integers $\forall i \in \{1,2,3\}$, with $t \in \mathbb{R}$ and $\mu > 0$. Let $(q_1,q_2,q_3) = \overline{q}$, $(\gamma_1,\gamma_2,\gamma_3) = \overline{\gamma}$, $(n_1,n_2,n_3) = \overline{n}$, and

$$Comp_{[\overline{q},\overline{n},\overline{\gamma},\mu]}(\cdot) = \left(q_1\frac{d}{dt} - \gamma_1\right)^{n_1} \left(q_2 I_a^{1-\mu} - \gamma_2\right)^{n_2} \left(q_3\frac{d}{dx} - \gamma_3\right)^{n_3}(\cdot). \quad (13)$$

For a function f that satisfies the necessary conditions for the existence of the operator in question, we have the following examples considering that $\gamma_1, \gamma_2, \gamma_3$ are functions that depends only in the the value of μ.

Example 2. Let $n_1 = n_2 = n_3 = 1$, then

$$Comp_{[\overline{q},\overline{n},\overline{\gamma},\mu]}(f) = \left(I_a^{1-\mu}h\right)\left[q_1q_2' - \gamma_1 q_2\right] + \left(D_a^{\mu}h\right)\left[q_1q_2\right] + \left[\gamma_1\gamma_2 h - q_1\gamma_2 h'\right].$$

Where $h = q_3 f' - \gamma_3 f$. ($'$ means the derivative with respect to the variable under consideration).

Example 3. Let $n_1 = 2, n_2 = n_3 = 1$ and let h be the same as before, then:

$$Comp_{[\overline{q},\overline{n},\overline{\gamma},\mu]}(f) = \left(I_a^{1-\mu}h\right)\left[(q_2'q_1)' - \gamma_1^2 q_2\right] + \left(D_a^{\mu}h\right)\left[(q_1q_2)' + q_2'q_1 - \gamma_1 q_2 + \gamma_1 q_1 q_2\right] + \left[q_1q_2\left(D_a^{\mu}h\right)' - \gamma_2(q_1h')' - \gamma_1^2\gamma_2 h\right].$$

In this work, it is important to mention that the associative property is assumed to be valid when applying composition. On the other hand, we also focus on the numerical implementation of the weighted q operator that is defined below using the Grünwald–Letnikov approach, which provides a direct discretization of fractional derivatives, enabling computational simulations in scenarios where exact analysis is unattainable.

4. THE OPERATOR ${}^{\overline{q}}D_x^{\mu}$

Definition 4. Let $q_1(t,\mu)$ be continuous and $q_2(t,\mu)$ continuously differentiable in t. For $\gamma_i = 0, n_2 = 1, n_3 = 0, n_1 = [\mu] + 1$, and $q_3 = 0$, $\forall i \in \{1,2,3\}$, let $Comp_{[\overline{q},\overline{n},\overline{\gamma},\mu]}(f)[t] = \left({}^{\overline{q}}D_x^{\mu} f\right)(t)$ denote the **$\overline{q}$-weighted fractional Riemann-Liouville derivative** of order $\mu > 0$. For $q_1, q_2 \in AC^s(\mathbb{R})$

$$\left({}^{\overline{q}}D_a^{\mu} h\right)(t) = \left(q_1(t,\mu)\frac{d}{dt}\right)^s q_2(t,\mu)\left(I_a^{s-\mu}h(x)\right)(xt, \quad (14)$$

where $s = [\mu] + 1, x > a$ and $[\mu]$ denotes the integer part of μ. See Contreras et al. (2025)

If $0 < \mu < 1$ and $\lim_{\mu\to 1} q_1(t,\mu) = \lim_{\mu\to 1} q_2(t,\mu) = 1$. we recover classical case.

Example 4. For convenience if we take $q_2(t,\mu) = (x-a)^{\mu-1}$ so then (14) takes the form for $0 < \mu < 1$.

$$\left({}^{\overline{q}}D_a^{\mu} h\right)(t) = q_1(t,\mu)\left(\frac{d}{dt}\right)(t-a)^{\mu-1}\left(I_a^{1-\mu}h\right)(t). \quad (15)$$

and $\left({}^{\overline{q}}D_a^{\mu} 1\right)(t) = 0$.

Example 5. Consider $q_1 = 2$, $q_2(x) = Cosh(x)$, $h(t) = 1$ and $\mu < 1$. then

$$\left({}^{\overline{q}}D_a^{\mu} 1\right)(t) = \frac{d}{dx}\left(Cosh(x)\left(I_a^{1-\mu}1\right)(x)\right) = \frac{2x^{-\mu}\cosh(x)}{\Gamma(1-\mu)} + \frac{2x^{1-\mu}\sinh(x)}{(1-\mu)\Gamma(1-\mu)}. \quad (16)$$

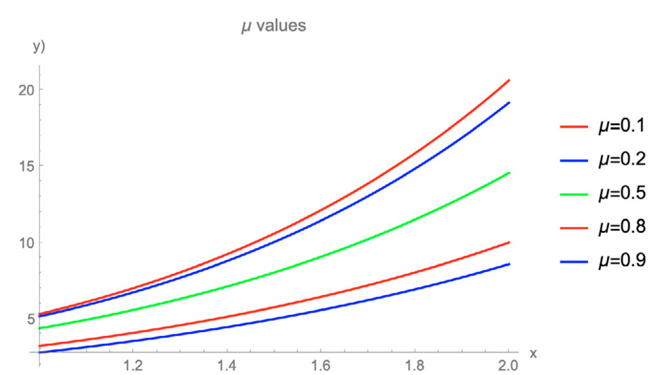

Fig. 1. Graph of the derivative of the function (16)

4.1 Some properties of the weighted operator

It is not difficult to see that the operator defined in (4) is linear. As in the case of the Riemann-Liouville operator, the semi-group property and the Leibniz rule for the weighted fractional operator are generally not satisfied. The following lemma proposes a property that is analogous to the former by combining the weighted fractional operator and the Riemann-Liouville derivative.

Lemma 3. Let μ, β be positive numbers such that $\mu + \beta < 1$, and $h(t) \in I_a^\beta(L_1)$. Then the property

$$^{\overline{q}}D_a^\mu(D_a^\beta h) = {}^{\overline{q}}D_a^{\mu+\beta} h. \tag{17}$$

holds almost everywhere on $[a, b]$.

Proof 1. Let μ, β be positive numbers such that $\mu+\beta < 1$, and $h(t) \in I_a^\beta(L_1)$. Then, using Lemma 2.3 and Lemma 2.5 from Kilbas et al. (2006),

$$^{\overline{q}}D_a^\mu(D_a^\beta h) = \left(q_1 \frac{d}{dt}\right)^s \left(q_2 \left(I_a^{s-\mu} D_a^\beta h\right)\right) = \left(q_1 \frac{d}{dt}\right)^s \left(q_2 \left(I_a^{s-\mu-\beta} I_a^\beta D_a^\beta h\right)\right) = \left(q_1 \frac{d}{dt}\right)^s \left(q_2 \left(I_a^{s-\mu-\beta} h\right)\right) = {}^{\overline{q}}D_a^{\mu+\beta} h.$$

Similarly, an analogous for the Leibniz rule for the weighted fractional derivative is presented by the following lemma.

Lemma 4. Let g, f be analytical real-valued functions and $\mu < 1$. Then $^{\overline{q}}D_a^\mu[f(t)g(t)] = q_1\left(\frac{d}{dt} q_2(t,\mu)\right) I_a^{1-\mu-k}(fg) + q_1 q_2 \sum_{k=0}^\infty \binom{\mu}{k} (D_a^k f)(D_a^{\mu-k} g)$.

Proof 2. The proof follows from direct calculations using (14) and (4).

Notice that this section has presented properties for D that hold for $\mu \in (0,1)$, but don't necessarily hold for higher-order weighted derivatives. Indeed, when considering $1 > \mu$, the composition $\left(q_1(t,\mu)\frac{d}{dt}\right)^s$, $\quad s = [\mu] + 1$, becomes more complicated due to the Leibniz rule and it expands into a sum that is dependent on q_1 and its derivatives, rapidly increasing the number of terms of the weighted fractional derivative depending on its order. This allows for more elaborate ways to modify the weighted operator using q_1 and q_2, which are worth exploring. For this paper, we will limit ourselves to $\mu \in (1, 2)$, as the study of (14) for an arbitrary $0 < \mu$ is outside the paper's scope. This section aims to present some of the properties that arise when $s = 2$ for the $\overline{q}$-weighted fractional Riemann-Liouville derivative.

Lemma 5. Let $\mu > 1$, and let f be a function as specified in the definition of the operator under consideration. If $q_1(t) = e^t$, $q_2(t) = e^{-t}$, then for $t > a$, the following holds:

$$\left(^{\overline{q}}D_a^\mu f\right)(t) = 0.$$

Proof 3. Let $s = [\mu] + 1 \geq 2$. Then, upon applying the operator, we find that the following expression is satisfied:

$$\left(^{\overline{q}}D_a^\mu f\right)(t) = \left(q_1 \frac{\partial}{\partial t}\right)^{s-2} \Bigg(\left(I_a^{s-\mu} f\right)(t)\left[q_1 q_1' q_2' + q_1^2 q_2''\right] + \left(\frac{d}{dt}\right)\left(I_a^{s-\mu} f\right)(t)\left[q_1 q_1' q_2 + q_1^2 q_2'\right] + \left(\frac{d}{dt}\right)^2 \left(I_a^{s-\mu} f\right)(t)\left[q_1^2 q_2' + q_1^2 q_2\right] \Bigg).$$

It is evident that the specified functions q_1 and q_2 satisfy the following system of ODEs simultaneously:

$$\begin{aligned}
&(1) \quad q_1(q_1' q_2' + q_1' q_2'') = 0 \Leftrightarrow \quad q_1' q_2' + q_1' q_2'' = 0,\\
&(2) \quad q_1(q_1' q_2 + q_1 q_2') = 0 \Leftrightarrow \quad q_1' q_2 + q_1 q_2' = 0,\\
&(3) \quad (q_1^2 q_2' + q_1^2 q_2) = 0 \Leftrightarrow \quad q_2' + q_2 = 0.
\end{aligned}$$

From which we obtain the desired result ($'$ means the derivative with respect to the variable under consideration).

In order to characterize the functions q_1 and q_2 such that the operator becomes zero in the case where $0 < \mu < 1$, we make use of the following lemma:

Lemma 6. Let $0 < \mu < 1$, and let $f = D_a^{1-\mu}(\frac{1}{q_2})$. If $\frac{1}{q_2} \in I_a^{1-\mu}(L_1)$, then $^{\overline{q}}D_a^\mu f = 0$.

Proof 4. The proof follows immediately from the fact that the expression defines a first-order ordinary differential equation in the function f, which can be solved using standard techniques from the theory of linear differential equations with integrating factors.

5. INCORPORATING FRACTIONAL PID CONTROL VIA THE $q-$OPERATOR

The composition operator introduced in this work provides a generalization of various fractional operators used in the development of FOPID controllers. In this section, we outline two main approaches for embedding PID-type structures within the *Comp* operator (13) formalism and discuss structural features that distinguish this approach from conventional FOPID controllers.

5.1 Intrinsic FOPID Structure

By definition, the operator (13) applied to a function f is the application of three operators to f. We can expand (where "expand" refers to something analogous to expanding a product) the definition of the operator applied to f, thereby obtaining a linear combination of well-known integro-differential operators, which naturally yields a structure reminiscent of classical FOPID controllers. Specifically, the expansion includes terms that can be interpreted as: a proportional component (order 0), a derivative component of fractional order μ, an integral component of fractional order $1 - \mu$ and some weighted terms, with $\mu \in (0, 1)$ representing the main fractional parameter of the operator.

Unlike traditional FOPID controllers, where the fractional orders of the integral and derivative terms are treated as independent tuning parameters, the composition operator intrinsically couples them. This structural coupling introduces a constraint on the parameter space of the controller. While this may reduce tuning flexibility, it also enforces a kind of internal symmetry that may affect the system in terms of robustness or interpretability.

5.2 Laplace-Domain Formulation via Convolution

An alternative approach involves reformulating the Composition operator (13) in the Laplace-domain. Since fractional differentiation generally breaks the multiplicative structure under the Laplace transform $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}L\{g(t)\}$ a viable strategy is to replace point-wise multiplication with convolution in the time domain

$f(t)g(t) \to (f*g)(t)$. Under this reinterpretation, the Composition operator can be preserved more faithfully when translated to the Laplace domain, potentially simplifying the analysis of linear systems and frequency response. However, this shift to a convolution-based formulation alters the original definition of the operator and alters its differential structure, so it becomes more attractive from the standpoint of computational implementation.
Moreover, in this convolutional interpretation, the composition operator no longer inherently carries proportional, integral, or derivative behavior in the classical sense. Instead, it can be treated as an independent dynamic term. This opens the possibility of using this operator itself as the derivative component of a new form of PID control, where the controller includes: a classical proportional gain, a fractional integral term, and a convolutional composition term.

6. THE WEIGHTED OPERATORS WITH CONVOLUTIONS AND ITS NUMERICAL IMPLEMENTATION USING GRÜNWALD–LETNIKOV

The main purpose of this section is to propose operators that can be implemented numerically, for example, in transform spaces. We have already made progress in this direction; see Gude et al. (2024b). The natural way to apply these operators in a transformed space is through convolution, due to the well-known property of the transform with respect to convolution; see Podlubny (1999b); Caponetto et al. (2010). Motivated by the operators presented in the previous sections, we aim to propose operators using the convolution operation. However, this results in complexity when trying to describe them in a general way using, for example, Kilbas et al. (2006); Samko et al. (1993). For this reason, we will focus only on two types, which are the ones we have worked with the most. (Due to space constraints, implementations of these operators are not presented.) $ConvComp_{[\overline{q},\overline{n},\overline{\gamma},\mu]} = \left(q_1 \frac{d}{dt} - \gamma_1\right)^{n_1} * \left(q_2 I_a^{1-\mu} - \gamma_2\right)^{n_2} * \left(q_3 \frac{d}{dt} - \gamma_3\right)^{n_3}$.
Where $*$ denote the convolution and The Grünwald-Letnikov derivative is defined as (See Palacios et al. (2023); Podlubny (1999a)):
Definition 7. Let $\mu > 0$, $f \in C^k[a,b]$, and $a < t \le b$. Then $\mathcal{G}_a^{\mu} f(t) = \lim_{N\to\infty} \frac{\Delta_{h_N}^{\mu} f(t)}{h_N^{\mu}} = \lim_{h\to 0} \frac{1}{h^{\beta}} \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} f(t - kh)$, with $h = \frac{(t-a)}{N}$, $N = 1, 2, \ldots$.

This definition involves a sum of values of the function $f(t)$ at different points.
Remark 8. Despite being expressed using the convolution symbol, the $ConvComp$ operator does not represent a triple convolution of operators. Instead, the convolution notation is employed to highlight that this operator is a variant of the previously defined (13), in which multiplication is selectively replaced by convolution depending on the context.
Remark 9. It is worth mentioning that the Grünwald-Letnikov derivative can be seen as a discrete approximation of the Riemann-Liouville derivative, and in the limit, as the step size goes to zero, the Grünwald-Letnikov derivative becomes a continuous fractional derivative.

Based on the operator (14), and considering numerical implementations with much greater simplicity and computational ease, we slightly modify the structure of the operator, keeping the form, but now thinking of it in terms of convolutions. $\left({}^{\overline{q}}D_a^{\mu} h\right)(t) = q_1(t,\mu) * \frac{d}{dt} q_2(t,\mu) * \left(I_a^{1-\mu} h\right) + q_1(t,\mu) * q_2(t,\mu) * D_a^{\mu} h$. Having the definition of the Grünwald-Letnikov, we can define our version of the weighted operator, which will be used for the numerical implementations in this work.

6.1 The Grünwald-Letnikov $\overline{q}$-weighted version

$\left({}^{\overline{q}}\mathcal{G}_t^{\mu} h\right)(t) = q_1(t,\mu) * \frac{d}{dt} q_2(t,\mu) * I_a^{1-\mu} h + q_1(t,\mu) * q_2(t,\mu) * \mathcal{G}_t^{\mu} h$.

Using the following theorem:
Theorem 1. Let f and g be functions such that $f \in C^1$ and $g \in L^1$, Then the derivative of the convolution satisfies: $\frac{d}{dt}(f*g)(t) = \left(\frac{df}{dt} * g\right)(t) = \left(f * \frac{dg}{dt}\right)(t)$ provided the relevant derivatives and integrals exist (See Folland (2009)) .

We obtain

$$\begin{aligned} \left({}^{\overline{q}}\mathcal{G}_t^{\mu} h\right)(t) &= \\ q_1(t,\mu) * q_2(t,\mu) * \mathcal{G}_t^{\mu} h(t) + q_1(t,\mu) * q_2(t,\mu) * \mathcal{G}_t^{\mu} h(t) &= \\ 2q_1(t,\mu) * q_2(t,\mu) * \mathcal{G}_t^{\mu} h(t). \end{aligned} \tag{18}$$

Remark 10. Note that, first, operator (6) is not the same as operator (14). Second, if either of the weights q_1 or q_2 are constants, the operator (18) reduces to:

$$\left({}^{\overline{q}}\mathcal{G}_t^{\mu} h\right)(t) = 2q_1 q_2(t,\mu) * \mathcal{G}_t^{\mu} h. \tag{19}$$

Example 6. Consider $q_1 = 2$, $q_2(t) = Cosh(x)$, $h(t) = 1$ and $\mu < 1$. then

$$\begin{aligned} \left({}^{\overline{q}}D_a^{\mu} h\right)(x) &= 2Cosh(x) * \left(I_a^{1-\mu} 1\right) = \\ &\frac{e^{-x} x^{-\mu}}{\Gamma(1-\mu)} (x E_{\mu}(-x) + e^{2x} x E_{\mu}(x) \\ &+\Gamma(1-\mu)\left((-x)^{\mu} - e^{2x} x^{\mu}\right)). \end{aligned} \tag{20}$$

This last resulting function is part of a causal function, and E_{μ} is the one-parameter Mittag-Leffler function; see Podlubny (1999a); Samko et al. (1993); Kilbas et al. (2006).

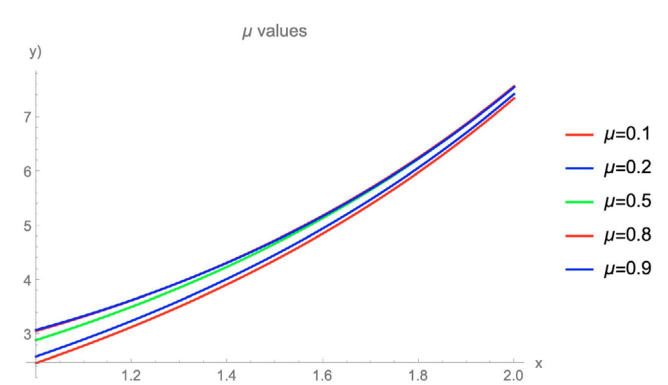

Fig. 2. Graph of the derivative of the function (20)

6.2 The Grünwald-Letnikov $\overline{q}, \gamma$-weighted version

Two composition Let $q_1, q_3, \gamma_1, \gamma_3 \in \mathbb{R}$ and the real valued function $q_2, \gamma_2 \in C^1(\Omega \subseteq \mathbb{R})$. (The functions can be dependent on the fractional parameter) Then the operator for $n_3 = 0$ and $n_1 = n_2 = 1$: $Comp_{[\overline{q},\overline{n},\overline{\gamma},\mu]} = \left(q_1 \frac{d}{dt} - \gamma_1\right)\left(q_2 I_a^{1-\mu} - \gamma_2\right) = q_1 \frac{d}{dt}(q_2) I_a^{1-\mu} + q_1 q_2 D_a^{\mu}$ $- q_1 \frac{d}{dt}\gamma_2 - \gamma_1 q_2 I_a^{1-\mu} + \gamma_1\gamma_2$.

We rewrite the expression using the convolution product symbol $*$ that gives: $Comp^*_{[\overline{q},\overline{n},\overline{\gamma},\mu]} = q_1 * (\frac{d}{dt} q_2) * I_a^{1-\mu} + q_1 * q_2 * D_a^{\mu} - q_1 * \frac{d}{dt}\gamma_2 - \gamma_1 * q_2 * I_a^{1-\mu} + \gamma_1 * \gamma_2 = 2q_1 * q_2 * D_a^{\mu} - \gamma_1 * q_2 * I_a^{1-\mu} + [-q_1' + \gamma_1] * \gamma_2$.

Three composition convolution version Let $q_1, q_3, \gamma_1, \gamma_3 \in \mathbb{R}$ and the real valued function $q_2, \gamma_2 \in C^1(\Omega \subseteq \mathbb{R})$. (The functions can be dependent on the fractional parameter) Then the operator for $n_1 = n_2 = n_3 = 1$ can be writing as: $ConvComp_{[\overline{q},\overline{n},\overline{\gamma},\mu]} = 2q_1q_2 * \mathcal{G}_t^{\mu}[q_3h' - \gamma_3 h] + \gamma_1 q_2 * I_a^{1-\mu}[\gamma_3 h - q_3 h'] + q_1\gamma_2' * [\gamma_3 h + q_3 h'] + (\gamma_1 q_3 \gamma_2 - q_1 q_3 \gamma_2') * h' + \gamma_1\gamma_2\gamma_3 * h$.

7. CONCLUSIONS

This study introduced a new class of weighted fractional operators, constructed through the composition of differential and integral operators, with the aim of expanding the flexibility and applicability of fractional calculus in engineering problems. The proposed operator, ${}^{\overline{q}}D_a^{\mu}$, generalizes traditional fractional derivatives while preserving critical properties such as linearity. Although it does not satisfy the semigroup property or the Leibniz rule in their classical forms, we demonstrated how analogous behaviors can be derived by combining it with the Riemann–Liouville formulation. A numerical representation based on the Grünwald–Letnikov approach was developed to enable discrete-time simulations, particularly useful when analytical solutions are not tractable. Finally, we examined the interaction between the weighted operator and Laplace/convolution frameworks, reinforcing the potential of this methodology for applications in control systems and signal processing.

As future work, the implementation of these operators in PID-type problems remains, for example, for which we have already made some progress that we hope to present in future publications

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